INSTITUTE OF ACTUARIES' TEXT BOOK.

PART I.—INTEREST.
(INCLUDING ANNUITIES CERTAIN.)

BY WILLIAM SUTTON, M.A.

INSTITUTE OF ACTUARIES'

TEXT-BOOK

OF THE

PRINCIPLES OF INTEREST

(INCLUDING ANNUITIES-CERTAIN),

LIFE ANNUITIES, AND ASSURANCES,

AND THEIR PRACTICAL APPLICATION.

PART I.

INTEREST (INCLUDING ANNUITIES-CERTAIN).

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PREFACE.

THE Council of the INSTITUTE OF ACTUARIES have thought it desirable to publish separately the First Part of the TEXT-BOOK, dealing with questions involving the Theory of Compound Interest only; and in doing so, have to express their regret that the Author, Mr. W. SUTTON, has found that the pressure of other engagements will prevent him from completing the Second Part, relating to Mortality, and to Annuities and Assurances dependent thereon. Other arrangements have, however, been made for the completion of the Work, and it is hoped that the remaining portion will be published at an early date.

INTRODUCTION BY THE AUTHOR.

In the preparation of this Work, it has been sought, as far as possible, to give a tolerably complete treatise on the important subject of the Theory of Compound Interest. From the nature of the subject, much original matter is hardly to be looked for; and the Author particularly desires it to be understood that he has, except in special cases, abstained from quoting his authorities, for two reasons—one of which is, that in many matters it has proved to be quite impracticable to quote the original; and the other that, by avoiding these references, which might or might not be correct, a great saving of space has been effected.

He has to express his obligations to Mr. T. G. Ackland and Mr. J. Heron Duncan for their assistance in examining proof-sheets, &c.; and he desires to tender his best thanks to Major-General Hannyngton and Mr. Peter Gray, for their several communications to be found on pp. 164 and 166 respectively.

W. S.

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PART I.

INTEREST

CHAPTER I.

INTEREST, AMOUNTS, PRESENT VALUES, AND DISCOUNT.

(1) WHEN a person possessed of capital is desirous to employ it productively, he may either enter upon some sort of commercial or industrial undertaking, or he may be satisfied with lending his capital to another person. In the first case, he will, if the undertaking prove successful, earn what is called profit; and in the latter case he will, among other things, stipulate for a fixed and definite consideration for the loan of his capital. This consideration is termed *interest.* It is not necessary that the capital lent, or that the interest received for lending it, should be in the shape of money. A loan may be in any commodity, and the interest for the loan may be in the same or any other commodity. For instance, a loan of land may be considered as a loan of capital invested in land, and the interest, here called rent, may be in money or corn, or even in services. But as in all transactions for value it is found expedient by the common consent of mankind to take money as the standard of value, it is usual to treat interest questions in the same way; although, as will be evident, this is not essential, all that is required being that the loan and the interest should be denoted by quantities that are homogeneous. **Explanation of the term Interest.**

(2) The interest to be charged in any case will depend upon the amount of the loan, in future called the principal, and upon

the length of time for which it is made; and in ordinary language the unit of principal is taken as 1 and the unit of time as a year. In this way, interest is said to be charged *at the rate* of five *per-cent*, or ·05 per unit per annum, or merely five per-cent, the unit of time being understood to be a year. In what follows, we propose to take 1 as the principal or unit bearing interest unless otherwise stated.

Amount to which principal will accumulate at end of a year at compound interest.

(3) Let us at present assume that interest on this 1 is to be charged at the rate i per annum. It will be noticed that nothing is here said as to when the interest is payable. Suppose, however, that the interest is payable at the end of every mth part of a year. Then at the end of the first mth part of a year, the interest due and payable being proportional to the time, is $\frac{i}{m}$. Now if this interest is not paid, the borrower will clearly have the use of it, and may be charged interest upon it at the same rate as upon the principal. On this assumption the capital bearing interest for the second mth part of a year will be $1+\frac{i}{m}$, and the interest payable at the end of the second mth part of a year will therefore be $\frac{i}{m}\left(1+\frac{i}{m}\right)$, the two together making $\left(1+\frac{i}{m}\right)^2$. Similarly, at the end of the third mth part of a year, the interest payable would be

$$\frac{i}{m}\left\{1+\frac{i}{m}+\frac{i}{m}\left(1+\frac{i}{m}\right)\right\}=\frac{i}{m}\left(1+\frac{i}{m}\right)^2,$$

and the capital outstanding $\left(1+\frac{i}{m}\right)^2$, the two together making $\left(1+\frac{i}{m}\right)^3$. And at the end of the last mth part of the year we should have

$$\text{Interest due and payable} = \frac{i}{m}\left(1+\frac{i}{m}\right)^{m-1}$$

$$\text{Capital outstanding} \qquad = \left(1+\frac{i}{m}\right)^{m-1}.$$

The two together make $\left(1+\frac{i}{m}\right)^m$. In other words, the original principal of 1, will have accumulated at the end of the year, by the operation of interest, to $\left(1+\frac{i}{m}\right)^m$, and the interest

earned by the original capital of 1 in the course of the year will consequently be $\left(1+\frac{i}{m}\right)^m-1$.

Interest thus calculated is called *compound interest.*

Distinction between nominal and effective rates of interest.

Under these circumstances, i is called the nominal rate of interest convertible m times a year, and $\left(1+\frac{i}{m}\right)^m-1$ is called the corresponding effective rate of interest, and may be denoted by $i^{(m)}$.

Example: If the nominal rate of interest be 5 per-cent, and interest be payable quarterly, then the sum to which 1 will amount at end of a year is given by $\left(1+\frac{\cdot 05}{4}\right)^4=(1\cdot 0125)^4$.

Now $\log 1\cdot 0125=\cdot 005395$;

$$\therefore \log (1\cdot 0125)^4=\cdot 021580$$
$$=\log 1\cdot 05095.$$

So that if 5 per-cent be the nominal rate payable quarterly, then the equivalent effective rate is $(1\cdot 0125)^4-1=\cdot 05095$, or $5\cdot 095$ per-cent.

(4) Since we have $\left(1+\frac{i}{m}\right)^m-1=i^{(m)}$

$$\therefore \left(1+\frac{i}{m}\right)^m=1+i^{(m)}$$

$$\therefore \left(1+\frac{i}{m}\right)=\{1+i^{(m)}\}^{\frac{1}{m}}$$

$$\left(1+\frac{i}{m}\right)^2=\{1+i^{(m)}\}^{\frac{2}{m}}$$

$$\&c. = \&c.$$

When, therefore, the effective rate of interest is i, we shall have

Compound interest for $\frac{1}{m}$th part of a year $=(1+i)^{\frac{1}{m}}-1$

" " $\frac{2}{m}$ parts " $=(1+i)^{\frac{2}{m}}-1$

" " $\frac{m-1}{m}$ " " $=(1+i)^{\frac{m-1}{m}}-1$

" " a year $=(1+i)-1$.

On the other hand, if i is the nominal rate of interest convertible m times in a year, then

$$\text{Compound interest for } \frac{1}{m}\text{th part of a year} = \left(1+\frac{i}{m}\right)-1$$

$$\text{,, ,, } \frac{2}{m} \text{ parts ,, } = \left(1+\frac{i}{m}\right)^2-1$$

$$\text{,, ,, } \frac{m-1}{m} \text{ ,, ,, } = \left(1+\frac{i}{m}\right)^{m-1}-1$$

$$\text{,, a year } = \left(1+\frac{i}{m}\right)^m-1.$$

It will be seen, therefore, that it is necessary in all interest questions to ascertain the exact conditions of any given case: in other words, when a rate of interest is given, is the rate a nominal or effective rate? and if the former, how many times in a year is interest to be convertible?

Example: Let the rate of interest be 5 per-cent. Then

(1) On the assumption that 5 per-cent is the effective rate of interest, we have

$$\text{Compound interest for } \tfrac{3}{4}\text{ths of a year} = (1{\cdot}05)^{\frac{3}{4}}-1$$
$$= {\cdot}0373.$$

(2) On the assumption that 5 per-cent is the nominal rate,

(*a*) Convertible yearly,

Compound interest for $\tfrac{3}{4}$ths of a year = same as above;

(*b*) Convertible half-yearly,

$$\text{Compound interest for } \tfrac{3}{4}\text{ths of a year} = \left(1+\frac{{\cdot}05}{2}\right)^{\frac{3}{2}}-1$$
$$= {\cdot}0377.$$

(*c*) Convertible quarterly,

$$\text{Compound interest for } \tfrac{3}{4}\text{ths of a year} = \left(1+\frac{{\cdot}05}{4}\right)^{3}-1$$
$$= {\cdot}0380$$

Amount to which principal will accumulate at end of a year at simple interest.

(5) If in Article (3), the interest not paid at the end of the first mth part of a year is not considered to bear interest, then the interest payable at the end of the second mth part of a year will be the same as for the first part—namely, $\frac{i}{m}$; and this, with the principal and unpaid interest, will amount to $1+\frac{i}{m}+\frac{i}{m}=1+\frac{2i}{m}$. Similarly, on the same assumption, at the end of the third interval, principal and interest will amount to $1+\frac{3i}{m}$, and at the end of the mth interval to $1+\frac{mi}{m}$, or $1+i$. So that the original principal of 1 will have accumulated at the end of the year by the operation of interest to $1+i$, the interest earned by the original capital of 1 in the course of the year being i.

Interest thus calculated is called *simple interest*.

In this case, it will be noticed that the nominal rate of interest and the effective rate of interest are the same.

Example: If the rate of simple interest be 5 per-cent, then 1 will amount in a year to 1·05, and this no matter how often interest is payable.

Difference between compound and simple interest for term not exceeding a year.

(6) Let now x be the nominal rate of interest when convertible m times a year, which corresponds to an effective rate of interest i, then we have

$$\left(1+\frac{x}{m}\right)^m=1+i$$

and
$$1+\frac{x}{m}=(1+i)^{\frac{1}{m}}$$

$$\left(1+\frac{x}{m}\right)^2=(1+i)^{\frac{2}{m}}$$

$$\&c. \quad = \quad \&c.$$

and generally
$$\left(1+\frac{x}{m}\right)^{m-1}=(1+i)^{\frac{m-1}{m}}.$$

Now when i is less than unity, the series obtained by the expansion of $(1+i)^{\frac{m-1}{m}}$ takes the form 1 + an infinite series of terms, of which the first term, viz., $\frac{m-1}{m}i$, is numerically the

greatest, and which are alternately positive and negative, and each term numerically less than the preceding term. The series is therefore convergent, and less than $\frac{m-1}{m}i$. *(See Algebra-Binomial Theorem and Convergency of Series.)*

Therefore $$(1+i)^{\frac{m-1}{m}} < 1 + \frac{m-1}{m}i,$$

and $$\left(1+\frac{x}{m}\right)^{m-1} - 1 < \frac{m-1}{m}i.$$

Now $\left(1+\frac{x}{m}\right)^{m-1} - 1$ denotes the interest accrued at the end of the $(m-1)$th interval, on the assumption of *compound interest at the nominal rate* x *convertible* m *times a year*, and $\frac{m-1}{m}i$ is the ***simple interest*** for the same period at the rate i. It will be noted that in the case of simple interest, the interest for any portion of a year is simply a proportionate part of the interest for a year on the original capital; whereas in the case of compound interest, interest is charged not only on the original principal but on the outstanding interest. As, however, the nominal rate of interest x employed in the case of compound interest is so taken that the corresponding effective rate is equal to the rate at which simple interest is charged, for all periods less than a year the compound interest is less than the simple interest.

If, on the other hand, the nominal rate employed in the case of compound interest be the same rate as that at which simple interest is charged, then, for every interval after the first, the compound interest will be greater than the simple interest.

Example: If the effective rate of interest be 5 per-cent, then the compound interest for $\frac{3}{4}$ths of a year is (Art. 4), $(1{\cdot}05)^{\frac{3}{4}}-1$ $=\cdot0373$, whereas the simple interest for the same time is $\frac{3}{4}$ of $\cdot05=\cdot0375$.

If the nominal rate be 5 per-cent convertible quarterly, then the compound interest and simple interest for $\frac{1}{4}$th of a year are each $=\frac{1}{4}$ of $\cdot05$
$=\cdot0125$.

The compound interest for $\frac{3}{4}$ths of a year $=\cdot0380$,
whereas the simple interest „ „ $=\cdot0375$,

(7) If in the expression $\left(1+\frac{i}{m}\right)^m$ we make $m=\infty$, then the limiting value of such expression, which may be denoted by $Lt\left(1+\frac{i}{m}\right)^m_{m=\infty}$, can be shown to be ϵ^i, where ϵ is the base of Napier's system of logarithms, and is equal to

Amount at compound interest when interest is convertible momently.

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\text{\&c.}=2\cdot718281828\ldots\ldots$$

(See Algebra, Exponential Series.)

Now it is clear that theoretically there is no restriction to the frequency with which interest may be convertible. If we suppose, taking x as the nominal rate of interest, that interest is convertible monthly, then in the above formula 1 would accumulate in a year to $\left(1+\frac{x}{12}\right)^{12}$, and if convertible daily to $\left(1+\frac{x}{365}\right)^{365}$. In the same way, on the assumption that interest is convertible momently, then 1 would accumulate to $Lt\left(1+\frac{x}{m}\right)^m_{m=\infty}$, *i.e.*, ϵ^x.

(8) To find, therefore, the nominal rate of interest convertible momently which is equivalent to an effective rate of interest i, we have the equation

Connection between nominal and effective rates of interest when interest is convertible momently.

$$\epsilon^x=1+i,$$

or

$$\begin{aligned} x&=\log_\epsilon(1+i)\,. \\ &=\log_{10}(1+i)\times\log_\epsilon 10 \\ &=\log_{10}(1+i)\times 2\cdot30258509 \end{aligned}$$

(Algebra, Logarithms.)

This value of x we shall denote by δ, and if the nominal rate of interest convertible momently is i, the equivalent effective rate of interest may be denoted by $\bar{i}$.

When interest is convertible momently, the nominal rate of interest is sometimes called the *force of interest* or *force of discount*, so that δ would denote the force of discount when the effective rate of interest is i.

The value of $\delta=\log_\epsilon(1+i)$ may be approximately found from the formula

Formula for approximating to value of force of interest

$$\delta=\tfrac{1}{2}\left(\frac{i}{1+i}+i\right) \text{ approximately.}$$

For we have $\frac{1}{2}\left(\frac{i}{1+i}+i\right)=\frac{1}{2}\{i(1+i)^{-1}+i\}$

$$=\frac{1}{2}\{i(1-i+\&c.)+i\}$$

$$=i-\frac{i^2}{2} \text{ approximately.}$$

But *(Algebra, Logarithmic Series)*

$$\log_{\epsilon}(1+i)=i-\frac{i^2}{2}+\ldots.$$

$$=i-\frac{i^2}{2} \text{ approximately.}$$

$$\therefore\quad \delta=\log_{\epsilon}(1+i)$$

$$=i-\frac{i^2}{2} \text{ approximately,}$$

$$=\frac{1}{2}\left(\frac{i}{1+i}+i\right) \quad ,,$$

The two following tables are calculated from the formula

$$1+i^{(m)}=\left(1+\frac{i}{m}\right)^m.$$

In Table I., the nominal rate of interest i, and m the number of times interest is convertible in a year, are the arguments, and $i^{(m)}$ the tabular result.

Thus, for example, if the nominal rate of interest be 5 per-cent, payable quarterly, then $i=\cdot05$, $m=4$, and from the table we see that the corresponding effective rate of interest, or $i^{(m)}$, is 5·094534 per-cent.

In Table II., the effective rate of interest $i^{(m)}$ and m are the arguments, and i, the nominal rate of interest, is the tabular result.

Thus, for example, if the effective rate of interest be 5 per-cent when interest at the nominal rate is payable quarterly, then $i^{(m)}=\cdot05$ and $m=4$, and from the table we see that i, the corresponding nominal rate of interest, is 4·908893 per-cent.

TABLE I.

Nominal Rate of Interest per-cent.	Equivalent Effective Rate of Interest per-cent when Interest is convertible m times in a Year.						Nominal Rate of Interest per-cent.
	$m=1$	$m=2$	$m=4$	$m=12$	$m=365$	$m=\infty$	
2	2·000000	2·010000	2·015050	2·018435	2·020078	2·020134	2
$2\frac{1}{2}$	2·500000	2·515625	2·523535	2·528846	2·531425	2·531512	$2\frac{1}{2}$
3	3·000000	3·022500	3·033919	3·041596	3·045325	3·045453	3
$3\frac{1}{2}$	3·500000	3·530625	3·546206	3·556695	3·561797	3·561971	$3\frac{1}{2}$
4	4·000000	4·040000	4·060401	4·074154	4·080849	4·081077	4
$4\frac{1}{2}$	4·500000	4·550625	4·576509	4·593983	4·602496	4·602786	$4\frac{1}{2}$
5	5·000000	5·062500	5·094534	5·116190	5·126750	5·127110	5
6	6·000000	6·090000	6·136355	6·167781	6·183131	6·183655	6
7	7·000000	7·122500	7·185903	7·229008	7·250099	7·250818	7

Table II.

Effective Rate of Interest per-cent.	Equivalent Nominal Rate of Interest per-cent when Interest is convertible m times in a year.						Effective Rate of Interest per-cent.
	$m=1$	$m=2$	$m=4$	$m=12$	$m=365$	$m=\infty$	
2	2·000000	1·990099	1·985172	1·981897	1·980316	1·980263	2
$2\frac{1}{2}$	2·500000	2·484567	2·476898	2·471798	2·469345	2·469261	$2\frac{1}{2}$
3	3·000000	2·977831	2·966828	2·959523	2·956000	2·955880	3
$3\frac{1}{2}$	3·500000	3·469900	3·454978	3·445075	3·440305	3·440143	$3\frac{1}{2}$
4	4·000000	3·960781	3·941362	3·928488	3·922282	3·922071	4
$4\frac{1}{2}$	4·500000	4·450483	4·425996	4·409768	4·401954	4·401689	$4\frac{1}{2}$
5	5·000000	4·939015	4·908893	4·888948	4·879343	4·879016	5
6	6·000000	5·912603	5·869538	5·841060	5·827356	5·826891	6
7	7·000000	6·881609	6·823409	6·784970	6·766492	6·765865	7

Amounts at compound and simple interest for term greater than a year.

(9) Hitherto the effect of interest has been considered for a term not exceeding a year, that being the unit of time used in defining the rate of interest; and it is now proposed to consider the effect when the time over which interest is to run is greater than such unit. Taking i as the nominal rate of interest, it has been shown that at simple interest a principal of 1 amounts at the end of a year to $1+i$, whereas at compound interest, interest being convertible at the end of every mth portion of a year, the amount at the end of a year is $1+i^{(m)}$, where $i^{(m)}=\left(1+\frac{i}{m}\right)^m-1$. Now at simple interest, where the original principal alone bears interest, the interest for the second year will be i, just as for the first year; and the total accumulation, or the amount, at the end of the second year, will be $1+i+i=1+2i$. But at compound interest, at the beginning of the second year, the principal bearing interest $=\left(1+\frac{i}{m}\right)^m$, and since, as has been already shown, a principal of 1 amounts at the end of a year to $\left(1+\frac{i}{m}\right)^m$, it follows that a principal of $\left(1+\frac{i}{m}\right)^m$ at the beginning of the second year will amount to $\left(1+\frac{i}{m}\right)^m \times \left(1+\frac{i}{m}\right)^m = \left(1+\frac{i}{m}\right)^{2m}$ at the end of that year. Similarly, at the end of the third year, the simple interest to be added will be i, and the total accumulation (principal + interest) or amount at the end of that year $1+i+i+i=1+3i$.

At compound interest, the total accumulation or amount will be

$$\left(1+\frac{i}{m}\right)^{2m} \times \left(1+\frac{i}{m}\right)^m = \left(1+\frac{i}{m}\right)^{3m}.$$

At the end of n years we shall have

At simple interest,

Total accumulation or amount $= 1+ni$,

At compound interest,

Total accumulation or amount $= \left(1+\frac{i}{m}\right)^{mn} = \{1+i^{(m)}\}^n$.

If in the formula $\left(1+\frac{i}{m}\right)^{mn}$ we assume m to be ∞, then we arrive at the conclusion that the amount to which a principal

of 1 will accumulate at the end of n years, on the assumption that the nominal rate of interest is i, and that interest is convertible momently, is

$$\mathrm{L}t\left(1+\frac{i}{m}\right)^{mn}_{m=\infty} = \left\{\mathrm{L}t\left(1+\frac{i}{m}\right)^{m}_{m=\infty}\right\}^{n}.$$

But $$\mathrm{L}t\left(1+\frac{i}{m}\right)^{m}_{m=\infty} = \epsilon^{i}, \quad \text{(Art. 7)}$$

$$\therefore \quad \mathrm{L}t\left(1+\frac{i}{m}\right)^{mn}_{m=\infty} = \epsilon^{in}.$$

Definition of Present Value.

(10) Instead of starting with a given principal, and calculating to what amount it will accumulate at a given rate of interest in a given time, we may assume the said amount as given, and calculate what must be the principal which would accumulate at the given rate of interest, and in the given time, to the said amount. This it will be seen is a question of simple proportion; for if in a given time and at a given rate of interest,

a principal of 1 amounts, say, to X,

then a principal of $\frac{1}{\mathrm{X}}$ will amount to 1;

and $\frac{1}{\mathrm{X}}$ is said to be the *present value* of 1 due a given time hence at a given rate of interest. If, therefore, the sum to be accumulated be taken as the unit of reference, the accumulated sum is called the *amount;* and if the accumulated sum is taken as the unit of reference, the sum to be accumulated is called the *present value.*

Formulas for Present Value.

(11) As shown above, if a principal of 1 amount to X in a given time at a given rate of interest, then the present value of 1 due the same time hence at the same rate of interest will be $\frac{1}{\mathrm{X}}$, and this relation is general. So that, as a matter of fact, the present value of 1 due a given time hence at a given rate of interest, is always the reciprocal of the amount to which 1 will accumulate in the given time at the given rate of interest. In symbols, if i be the nominal rate of interest and m the number of times in a year interest is convertible in a year, then the present value of 1 due n years hence is

$$\frac{1}{\left(1+\frac{i}{m}\right)^{mn}}=\left(1+\frac{i}{m}\right)^{-mn}$$

$$=\{1+i^{(m)}\}^{-n}$$

When $m=\infty$, this becomes $\quad =\epsilon^{-in}$.

Similarly at simple interest the present value of 1 due n years hence will be denoted by

$$\frac{1}{1+ni}.$$

Example: The present value of 1 due 10 years hence, the nominal rate of interest being 5 per-cent, convertible quarterly, is given by the expression

$$\frac{1}{\left(1+\frac{\cdot 05}{4}\right)^{40}}=\frac{1}{(1\cdot 0125)^{40}}.$$

Now $\qquad \log 1\cdot 0125=\cdot 005395$;

$$\therefore \quad \log (1\cdot 0125)^{-40}=-\cdot 21580,$$
$$=\bar{1}\cdot 78420$$
$$=\log \cdot 60842.$$

Or this could have been calculated by means of Table I., which gives ·05094534 as the effective rate, equivalent to a nominal rate of 5 per-cent convertible quarterly.

Thus $\qquad$ present value $=(1\cdot 05094534)^{-10}$.

Now $\qquad \log 1\cdot 05095=\cdot 021580$,

$$\therefore \quad \log (1\cdot 05095)^{-10}=-\cdot 21580$$
$$=\bar{1}\cdot 78420$$
$$=\log \cdot 60842 \text{ as before.}$$

Compound Interest Tables of Amounts and Present Values.

(12) Compound Interest tables, therefore, naturally resolve themselves into two kinds—(1) Tables of Amounts; and (2) Tables of Present Values—the arguments in each case being the given rate of interest and the given time. In accordance with usage, a year is taken as the unit of time, but it is clear that there is no necessity for this, the only essential being that the rate of

interest argument shall correspond to the period of time taken as the unit. For example, in the general formula for amount

$$\left(1+\frac{i}{m}\right)^{mn}$$

the period of time taken as the unit would be $\frac{1}{m}$th of a year, and the rate of interest argument $\frac{i}{m}$, and then under the time argument mn we should have as our tabular result $\left(1+\frac{i}{m}\right)^{mn}$. Similarly for present value $\left(1+\frac{i}{m}\right)^{-mn}$.

Example: The following table gives the amounts and present values of 1 for a number of intervals of time, the rate of interest being 1 per-cent *per interval*:—

No. of Intervals (n)	Amount $=(1{\cdot}01)^n$	Present Value $=(1{\cdot}01)^{-n}$
1	1·010000	·990099
2	1·020100	·980296
3	1·030301	·970590
4	1·040604	·960980
5	1·051010	·951466
6	1·061520	·942045
7	1·072135	·932718
8	1·082856	·923483
9	1·093685	·914340
10	1·104622	·905287
11	1·115668	·896324
12	1·126825	·887449

Now $(1{\cdot}01)^n=\left(1+\frac{m\times{\cdot}01}{m}\right)^{m\times\frac{n}{m}}$, so that the value of $(1{\cdot}01)^n$ in the above table gives us the amount to which 1 will accumulate in the time $\frac{n}{m}$ at the rate of interest $m\times{\cdot}01$ convertible m times a year. For instance, if $m=2$, and $n=6$, we learn that the amount to which 1 will accumulate in 3 years, the rate of interest being 2 per-cent payable half-yearly, is 1·061520. In the same way, if $m=4$, the amount to which 1 will accumulate in 3 years, the rate of interest being 4 per-cent convertible quarterly, is 1·126825. Similarly for present values.

(13) When a sum is payable at some future time, we have seen that its present value is such a principal as, when put out to interest for the given time at a given rate of interest, will amount to the given sum at the end of the given time. The difference between the given sum and its present value is called the discount, and will be given by the general formula. Discount on sum payable at some future time = sum payable less present value of sum payable. **Explanation of term Discount.**

(14) Taking 1 as the given sum, n as the number of years before it is due, i as the nominal rate of interest, and m as the number of times in a year interest is convertible, then **Formulas for Discount.**

$$\text{discount} = 1 - \left(1 + \frac{i}{m}\right)^{-nm}.$$

If i is taken as the effective rate of interest, so that interest is only convertible once a year ($m=1$), then if the number of years $=1$, we have

$$\begin{aligned}\text{discount} &= 1-(1+i)^{-1}\\ &= \frac{i}{1+i}.\end{aligned}$$

And if we denote $(1+i)^{-1}$ by v, we have discount $=1-v=iv$, which will therefore be the discount on 1 due a year hence, when the effective rate of interest is i. This value is, for convenience, frequently denoted by the symbol d.

Since the discount for a year $= iv$, we see that it is equal to the interest for a year, not upon the original principal 1, but upon its present value v; and this suggests another way of considering discount—namely, as the accumulated interest on the present value of the principal. Thus the interest accumulated on a principal of 1 at the end of n years being $(1+i)^n-1$, and the present value of the said principal being $(1+i)^{-n}$ or v^n, the discount on the said principal would be $\{(1+i)^n-1\} \times v^n = 1-v^n$.

At simple interest—that is, where interest is only charged on the original principal—the discount on a sum 1 due n years hence, would be

$$\begin{aligned}&1-\text{present value of } 1\\ &=1-(1+ni)^{-1}\\ &=\frac{ni}{1+ni}.\end{aligned}$$

In ordinary trading, the method of calculating discount is neither of the above, the custom being, if n be the number of years hence the principal of 1 is due, and i the rate of interest, to consider ni as the "discount." This is equivalent, it will be seen, to considering discount as denoting, not the simple interest on the present value of the principal, but as denoting the simple interest on the principal itself, so that when ni is greater than 1, the trade discount would be greater than the sum itself.

Comparison of various methods of calculating Discount.

(15) Thus, taking i as the annual effective rate of interest, we have the discount on a principal of 1 due n years hence denoted

(1) When taking compound interest, by $1-(1+i)^{-n}=\dfrac{(1+i)^n-1}{(1+i)^n}$,

(2) „ simple „ by $1-(1+ni)^{-1}=\dfrac{ni}{1+ni}$,

(3) In trade by ni.

Example: Let 1 be the sum due 10 years hence at 6 per-cent interest, then

(1) Discount at compound interest $=1-(1{\cdot}06)^{-10}$
$=\cdot 44161$

(2) Discount at simple interest $=\dfrac{\cdot 6}{1{\cdot}6}$
$=\cdot 37500$

(3) Trade discount $=\cdot 6$.

The incorrectness of the method of calculating discount used in ordinary trading is easily shown by taking the extreme case of $n=\infty$, that is, where the principal is due an infinitely long time hence. In both (1) and (2) the discount becomes equal to the principal itself, but in (3) the discount becomes an infinitely large quantity.

Geometrical illustrations of simple and compound interest and discount.

(16) A general illustration of the formulas for amounts and present values may be shown as follows:—

Taking a system of co-ordinate axes OX and OY (Figure 1), let us take OS on the axis OY as equal to 1, the principal, and let us take on the axis OX the lines OA_1, OA_2, OA_3 . . . to the right of OY, &c., and OV_1, OV_2, OV_3 . . . to the left, to denote 1 year, 2 years, 3 years, &c., respectively. Then if we take ordinates A_1P_1, A_2P_2, A_3P_3, &c., equal to $1+i$, $(1+i)^2$, $(1+i)^3$,

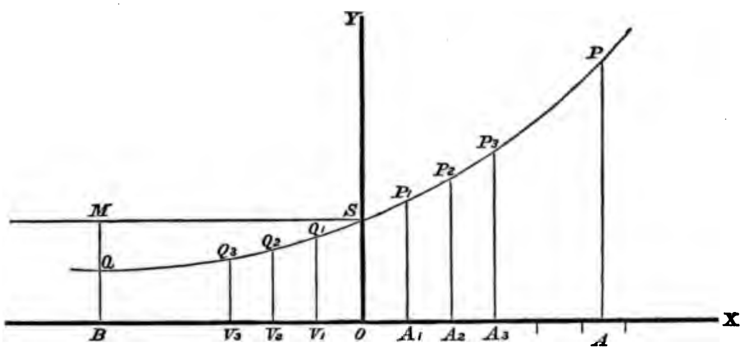
&c., to the right of OY, and ordinates V_1Q_1, V_2Q_2, V_3Q_3, &c., ordinates to the left, equal $(1+i)^{-1}$, $(1+i)^{-2}$, $(1+i)^{-3}$, &c., respectively, the series of ordinates to the right will represent the amount of 1 in 1 year, 2 years, 3 years, &c., respectively, at the rate of interest i; and the series to the left will represent the present value of 1 due at the end of 1 year, 2 years, 3 years, &c., respectively, at the same rate of interest.

The points P_1, P_2, P_3, &c., and Q_1, Q_2, Q_3 . . &c., are all points on the curve whose equation is

$$y=(1+i)^x,$$

commonly known as the Logarithmic Curve, whose property is that equidistant ordinates are in geometric progression.

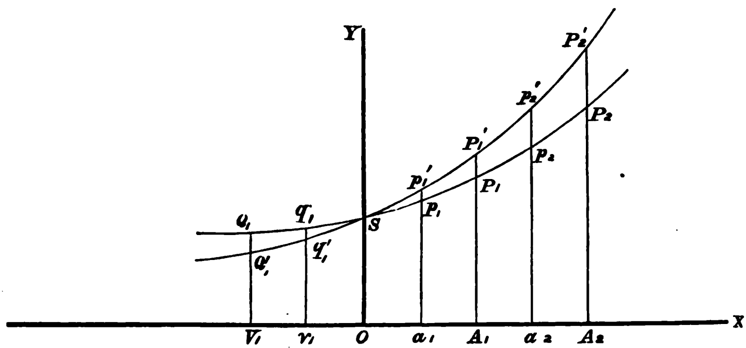
Figure 1.



If on the axis of x we measure off to the right of OY a line OA to denote any interval of time, whether consisting of an integral number of years or a fractional number, then the ordinate PA will denote the amount of 1 in the given time at the rate of interest i. Similarly, if OB be measured off to the left of OY, and equal in length to OA, then QB will denote the present value of 1 in the given time; and if a line be drawn through S parallel to OX, and cutting QB produced in M, then MQ denotes the discount corresponding to such present value.

Again (Figure 2), if we take $Oa_1=Ov_1=\frac{1}{2}$, then the ordinates p_1a_1 and q_1v_1 would respectively denote $(1+i)^{\frac{1}{2}}$ and $(1+i)^{-\frac{1}{2}}$, that is, the amount and present value of 1 for a period of half-a-year at the effective rate of interest i.

(Figure 2).



If, however, $a_1 p_1$ be produced to p'_1, so that $a_1 p'_1 = 1 + \frac{i}{2}$, and if the other ordinates be produced so as to be equal to $\left(1+\frac{i}{2}\right)^2$, $\left(1+\frac{i}{2}\right)^3$, $\left(1+\frac{i}{4}\right)^4$, &c., respectively, then another logarithmic curve would be formed, whose ordinates to the right of OY would denote the amount of 1 in the time denoted by the corresponding abscissa, at the rate of interest i convertible half-yearly. Similarly, the ordinates to the left of the axis OY would denote the present value of 1 corresponding to the time represented by the corresponding abscissæ, interest at the nominal rate i convertible half-yearly.

The equation to this second curve would be

$$y = \left(1+\frac{i}{2}\right)^{2x}.$$

In a similar way, any number of curves might be drawn, with similar properties, the general form of the equation being

$$y = \left(1+\frac{i}{m}\right)^{mx},$$

where m is a positive integer.

If in the equation $y = \left(1+\frac{i}{m}\right)^{mx}$ we take m to be infinite,—that is, assume interest at the nominal rate i to be convertible momently,—then, since we know that the limiting value of

$\left(1+\frac{i}{m}\right)^m_{m=\infty}$ is ϵ^i, we get as the equation to the curve in this case

$$y=\epsilon^{ix}.$$

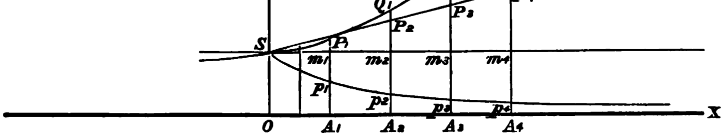
Giving to x any positive value, the corresponding value of y would be the sum to which a principal of 1 would amount in the time x, interest being convertible momently at the nominal rate i; and the present value of 1 due at the time x hence would be the reciprocal of this value of y or ϵ^{-ix}.

Again, at simple interest (Figure 3), taking, as before, a system of co-ordinate axes OX and OY, let OA_1, OA_2, &c., measured along the axis of x, denote intervals of 1 year, 2 years, &c.; then the formula for the amount of a principal of 1 at the end of x years, i being the rate of interest, is, if we call this amount y,

$$y=1+ix.$$

This is the equation of a straight line cutting the axis OY at a distance $1=OS$ from O, and inclined to the axis OX at an angle $=\mathrm{T}an^{-1}i$, that is, an angle whose tangent $=i$.

(*Figure* 3).



The ordinates P_1A_1, P_2A_2, &c., drawn as in the figure, will respectively be equal to $1+i$, $1+2i$, &c., or the amount of 1 at the end of 1 year, 2 years, &c., respectively.

Again, the present value of a principal of 1 due x years hence would be given by y in the equation

$$y=\frac{1}{1+ix}.$$

This is the equation to a hyperbola, of which OX is one of the asymptotes, that is, touches the curve at an infinite distance from the origin O, and is denoted in the figure as regards the part for positive values of x by the curve $Sp_1p_2p_3p_4$ the ordinates p_1A_1, p_2A_2, &c., being respectively equal to $\frac{1}{1+i}$, $\frac{1}{1+2i}$, &c.

If we draw a line through S parallel to the axis OX, and intersecting $P_1\,A_1$, $P_2\,A_2$, &c. in m_1, m_2, &c. respectively, then the discounts at simple interest for 1 year, 2 years, &c. are respectively denoted by $m_1\,p_1$, $m_2\,p_2$, &c., and the commercial discount for the same terms by $P_1\,m_1$, $P_2\,m_2$, &c.

Again, taking the curved line passing through S and P_1 in the figure to denote the logarithmic curve $y=(1+i)^x$, it will be at once seen that for terms less than 1 year the compound interest is less than the simple interest for the same term, but greater than the simple interest afterwards by the quantities denoted by the lines $Q_2\,P_2$, $Q_3\,P_3$, &c.

On the determination of the quantities involved in the formulas for Amounts and Present Values.

(17) Taking the general formulas $y=\left(1+\frac{i}{m}\right)^{mn}$ and $y=\left(1+\frac{i}{m}\right)^{-mn}$ for amounts and present values respectively at compound interest, we see that there are four quantities involved, y, i, m, and n; so that, given any three of those quantities, it might be inferred that the fourth can always be obtained. There is one case, however, when this can only be approximately done, namely, in the case where y, i, and n are given, and it is required to find m. In this case the equation takes the form,

$$\text{Constant } =\phi(x)^x.$$

For if y, i, and n are given, and we call x the required value of m, we have

$$\text{Known quantity, or constant}=\left(1+\frac{i}{x}\right)^{nx}$$
$$=\phi(x)^x,$$

if we denote $\left(1+\frac{i}{x}\right)^n$ by $\phi(x)$, a function of n whose form is known. This equation cannot be directly solved; and it is proposed to consider it further on, in dealing with another part of the subject.

With the other cases there is no difficulty; for we have

(1) i, m, and n given, $$y=\left(1+\frac{i}{m}\right)^{mn},$$

(2) i, m, and y given, $$n=\frac{\log y}{m\log\left(1+\frac{i}{m}\right)},$$

(3) m, n, and y given, $$\log\left(1+\frac{i}{m}\right)=\frac{\log y}{mn};$$

$$\therefore\quad i=m\left\{\log^{-1}\left(\frac{\log y}{mn}\right)-1\right\}.$$

Example: Let $y=1{\cdot}126825$,

$n=3$,

$m=4$,

$i=\cdot04$,

the above formulas would give

(1) n, m, and i given: $y=(1{\cdot}01)^{12}$;

$$\therefore\quad \log y=12\log 1{\cdot}01$$
$$=12\times\cdot0043214=\cdot0518568;$$
$$\therefore\quad y=1{\cdot}126825.$$

(2) y, m, and i given: $$n=\frac{\log 1{\cdot}126825}{4\log 1{\cdot}01}$$
$$=\frac{\cdot0518568}{\cdot0172856}$$
$$=3.$$

(3) y, m, and n given: $$i=4\left\{\log^{-1}\left(\frac{\log 1{\cdot}126825}{12}\right)-1\right\}$$
$$=4\left\{\log^{-1}\left(\frac{\cdot0518568}{12}\right)-1\right\}$$
$$=4(\log^{-1}\cdot0043214-1)$$
$$=4(1{\cdot}01-1)$$
$$=\cdot04.$$

(For further Illustrations of this Chapter, see Chapter V.)

CHAPTER II.

ANNUITIES CERTAIN.

INTEREST CONVERTIBLE YEARLY AND ANNUITY PAYABLE YEARLY.

General explanation of Annuities.

(18) The word annuity is employed to denote a series of payments at stated periods or intervals of time, the said payments not necessarily being of the same magnitude. Annuities are of two kinds, those which are for certain fixed periods, and those which are dependent on some contingency—such, for instance, as the existence of a particular life. It is the former kind only that is now to be considered.

When the interval of time which has to elapse before the first payment becomes due is greater than the stated interval between the payments, the annuity is said to be deferred; and when not greater, the annuity is said to be in possession, or to begin to accrue, and is called an immediate annuity. Unless otherwise stated, the interval of time to elapse before the first payment falls to be made will be understood to be equal to the interval between the payments.

The magnitude of the periodic payments is generally described by stating how much is payable in the course of a year, that being the unit of time, and this should always in strictness be accompanied by a statement as to how often during the year the payments are to be made. Thus an annuity of 1 payable half-yearly, or by half-yearly instalments, would denote a series of payments of $\frac{1}{2}$ each, the first payment to be made a half year hence, and the following payments at the end of the succeeding half years. An annuity of 1 payable half-yearly, deferred 5 years, would denote a series of payments of $\frac{1}{2}$ each, the first payment due, not 5 years

hence, but $5\frac{1}{2}$ years hence, the time it is stated to be deferred having reference to the interval which must elapse before the annuity begins to accrue and comes into possession, or becomes an immediate annuity. An annuity-due is an annuity where the first payment is at once due and payable, so that an annuity of 1 payable half-yearly deferred 5 years may be otherwise described as an annuity due deferred $5\frac{1}{2}$ years.

Amounts and present values of Annuities.

As it is always assumed that money is bearing interest, it is clear that if the periodic payments constituting a given annuity are not paid as they become due, but are, for instance, all paid at a given date, then, if the given date coincides with that on which the last payment becomes due, all the preceding instalments will fall to be paid with interest to that date from the time when they respectively became due. The sum thus payable is called the amount of the given annuity. If, on the other hand, the given date coincides with the date from which the annuity runs or begins to accrue, or with any previous date, then all the instalments will fall to be discounted, and the sum thus payable is called the value at the given time of the given annuity. When the present time is taken as that from which discount is computed, the sum payable is called the present value of the annuity.

This process is called capitalization, and thus we have the capitalized amount, and the capitalized value of a given annuity.

It is usual to speak of the present value of an annuity as equal to so many years' purchase,—that is, so many times the amount receivable in respect of the annuity in one year. Thus, if an annuity where the amount receivable yearly in respect thereof is £100, be estimated to be of the present value £2,725, the said annuity may be described as worth $27\frac{1}{4}$ years' purchase. As we shall generally consider 1 as the amount receivable yearly in respect of the annuity, the present value of the annuity will, in all cases, give at once the number of years' purchase. Thus, if the value of an annuity of 1 for a given number of years at a given rate of interest be, say X, the said annuity may be described as worth X years' purchase.

Amount of Annuity at simple interest

(19) Let the annuity under consideration be one where the sum 1 is payable at the end of each year for n years from the present time, and let us first take the case where simple interest is allowed at the rate i per annum.

At the end of n years the payment due	1 year	hence will amount to	$1+(n-1)$
„ „	2 years	„	$1+(n-2)$
	. . . . .		. . . . .
„ „	$n-1$ years	„	$1+i$
„ „	n „	„	1

and the capitalized amount of the annuity—or, briefly, the amount of the annuity—at the end of the n years will be the sum of the series

$$1+(1+i)+(1+2i)+\ldots+\{1+(n-2)i\}+\{1+(n-1)i\},$$

an arithmetical series whose sum $=n+\dfrac{n.\overline{n-1}}{2}i.$

Present value of Annuity at simple interest.

Similarly, proceeding to find the capitalized present value—or, briefly, the present value—of the annuity, we have

Present value of the payment due 1 year hence $=\dfrac{1}{1+i}$ (Art. 11)

„ „	2 years	„	$=\dfrac{1}{1+2i}$
„ „	3 „	„	$=\dfrac{1}{1+3i}$
„ „	$n-1$ „	„	$=\dfrac{1}{1+(n-1)i}$
„ „	n „	„	$=\dfrac{1}{1+ni}.$

Hence the present value of the given annuity is the sum of the series

$$\frac{1}{1+i}+\frac{1}{1+2i}+\ldots\quad+\frac{1}{1+(n-1)i}+\frac{1}{1+ni}.$$

Difficulties attending the assumption of simple interest in the theory of Annuities Certain.

(20) No convenient formula for the summation of this series is obtainable.* If, however, we consider that where there are two annuities of exactly the same kind, both as to magnitude and duration; and that in the one case the payments are allowed to stand over at interest until the last becomes payable, and in the other case the payments are in the first instance discounted, and their present values then put out at interest at the same rate until the end of the term for which the annuity runs, the two

* *De Morgan has given a formula which will be found in Chapter VI., Art. (75).*

annuities should amount to the same, we ought to have the means of summing the above series. Adopting this line of reasoning, we have

Discounted value of first payment $=\dfrac{1}{1+i}$,

and this will amount at end of n years to $\dfrac{1+ni}{1+i}$.

Similarly, the discounted value of second payment $=\dfrac{1}{1+2i}$,

and this will amount at end of n years to $\dfrac{1+ni}{1+2i}$,

and so on, the discounted value of nth payment being $\dfrac{1}{1+ni}$, and its amount at the end of n years $\dfrac{1+ni}{1+ni}$, or 1. Thus the sum to which the discounted values will amount at the end of n years will be denoted by the sum of the series

$$\frac{1+ni}{1+i}+\frac{1+ni}{1+2i}+\dots\quad+\frac{1+ni}{1+n(-1)i}+\frac{1+ni}{1+ni}$$

$$=(1+ni)\left\{\frac{1}{1+i}+\frac{1}{1+2i}+\dots\quad+\frac{1}{1+(n-1)i}+\frac{1}{1+ni}\right\}$$

$=(1+ni)\mathrm{X}$, say.

But the sum of this series is by hypothesis to be the same as that of the annuity which has been allowed to accumulate. Thus we have

$$(1+ni)\mathrm{X}=n+\frac{n.(n-1)}{2}i \qquad \text{(Art. 19)}$$

$$\therefore\quad \mathrm{X}=\left\{n+\frac{n.(n-1)}{2}i\right\}\frac{1}{1+ni}.$$

On trial the value given by this formula will be found too great, and it is clear that the theory of simple interest is not consistent with the principle of capitalization in its accepted form. Various attempts have been made to reconcile them, but the following considerations will be sufficient to show the difficulty involved. If the sum 1 is accumulated at simple interest for $n-1$ years, it will amount to $1+(n-1)i$. If we call this S, and accu-

mulate S for another year, the amount at the end of that year, being n years from the commencement, will be $S(1+i)$.

Now
$$S(1+i)=\{1+(n-1)i\}(1+i),$$
$$=1+ni+(n-1)i^2.$$

But by the theory of simple interest, the sum to which 1 will amount in n years is $1+ni$, and we see that the difference between $1+ni+(n-1)i^2$ and $1+ni$ is $(n-1)i^2$, or the interest for one year on the interest accumulated at the end of $n-1$ years. In other words, simple interest never allows any sum accrued or due as interest to be hereafter considered as capital bearing interest—that is, does not allow interest to be capitalized. It is for this reason that, in all questions involving the capitalization of annuities, the theory of simple interest is in practice never employed, and it is not proposed to make any further use of this theory in what follows.

Amount of Annuity at compound interest.

(21) We now proceed to treat of annuities at compound interest. Let us first take the simple case of an annual payment of 1 for n years, the rate of interest assumed being i.

To find what the annual payments will amount to at the end of the n years by the operation of compound interest we proceed as follows:

The first payment of 1 being payable one year hence will be bearing interest for $n-1$ years, and will therefore, since interest at the rate i is convertible once a year, amount at the end of n years (Art. 9) to $(1+i)^{n-1}$

the second payment will amount at the end of n years to $(1+i)^{n-2}$

,, ,, ,, ,,

,, $(n-1)$th payment will amount at the ,, ,, $(1+i)$

,, nth ,, ,, ,, 1.

The total sum therefore to which the annual payments will amount at the end of n years will be the sum of the series

$$1+(1+i)+(1+i)^2+\ldots\ldots+(1+i)^{n-1}.$$

This is a geometric series whose sum is known (*Algebra-Geometric Series*) to be

$$\frac{(1+i)^n-1}{(1+i)-1}=\frac{(1+i)^n-1}{i}.$$

Present value of Annuity at compound interest.

(22) As it is clear that the present value of the given annuity ought to be such a sum as invested at interest will, at the end of the n years, amount to the same sum as the annual payments would together amount to at the end of the n years, if they respectively bear interest from the time of falling due, we have

$$\text{Present value of annuity} \times (1+i)^n = \text{amount of annuity}$$
$$= \frac{(1+i)^n - 1}{i};$$

$$\therefore \quad \text{Present value of annuity} = \frac{1}{(1+i)^n} \times \frac{(1+i)^n - 1}{i}$$
$$= \frac{1-(1+i)^{-n}}{i}.$$

This result would follow at once, as the sum of the geometric series representing the present values of the respective payments, namely, (Art. 10),

$$\frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \dots + \frac{1}{(1+i)^n}.$$

Let us denote $\frac{1}{1+i}$ by the symbol v, then $\frac{1}{(1+i)^n} = v^n$,

and the formula for the present value of the annuity is $\frac{1-v^n}{i}$, which we will denote by $a_{\overline{n}|}$.

It will be desirable to give some consideration to the formulas just obtained.

Another method for deducing the formulas for annuities.

(23) Another way of deducing the formula in question is as follows. The sum 1 will produce an annual payment of i, being the interest thereon for a year, so long as no portion of the said sum is repaid. Thus it is clear that a capital of 1 is equivalent in value to an annuity of i for n years, together with the sum now necessary, when put out at interest, to amount to 1 at the end of n years: that is,

$$1 = ia_{\overline{n}|} + v^n;$$

$$\therefore \quad a_{\overline{n}|} = \frac{1-v^n}{i}.$$

Similar reasoning will give the formula for the amount of an annuity. For as 1, by the operation of interest, amounts to $(1+i)^n$ at the end of n years, we have

Amount of an annuity of i for n years + original capital of $1=(1+i)^n$;

$\therefore$ Amount of an annuity of i for n years $=(1+i)^n-1$;

$\therefore$ Amount of an annuity of 1 for n years $=\dfrac{(1+i)^n-1}{i}$.

This process of reasoning will be found extremely useful in the deduction of various formulas.

The present value of an annuity if invested at same rate of interest as the annuity is calculated, will exactly provide the annual payments as they become due.

(24) Let us, as before, denote the present value of an annuity certain for n years, *i.e.*, $\dfrac{1-v^n}{i}$, by the symbol $a_{\overline{n}|}$;

so that $$a_{\overline{n}|}=\frac{1-v^n}{i}=v+v^2+\,.\,.\,+v^n;$$

$\therefore$ at the end of one year the amount to which $a_{\overline{n}|}$ would accumulate at interest is

$$(1+i)a_{\overline{n}|}=(1+i)\{v+v^2+\,.\,.\,+v^n\}$$
$$=1+v+v^2+\,.\,.\,+v^{n-1}.$$

Thus at the end of the first year there would be in hand the first annual payment of 1 of the given annuity, and in addition the sum

$$v+v^2+\,.\,.\,+v^{n-1}=a_{\overline{n-1}|}.$$

Similarly $$(1+i)a_{\overline{n-1}|}=1+v+v^2+\,.\,.\,+v^{n-2};$$

so that at end of second year there would be in hand the second payment of 1, and in addition the sum

$$v+v^2+\,.\,.\,+v^{n-2}=a_{\overline{n-2}|}.$$

Similarly, at the end of t years there would be in hand the tth payment of 1, and in addition the sum

$$v+v^2+\,.\,.\,+v^{n-t}=a_{\overline{n-t}|}.$$

At the end of the $(n-1)$th year, there would be in hand the $(n-1)$th payment of 1, and in addition the sum v, which, accumulated for the last or nth year, will provide the last annual payment of 1.

Thus we see that if a person entitled to an annual payment of 1 at the end of each year for n years accepts in lieu thereof a sum down $=\dfrac{1-v^n}{i}$, and invests this sum at the same rate of interest i, it will exactly provide the annual payments as they become due.

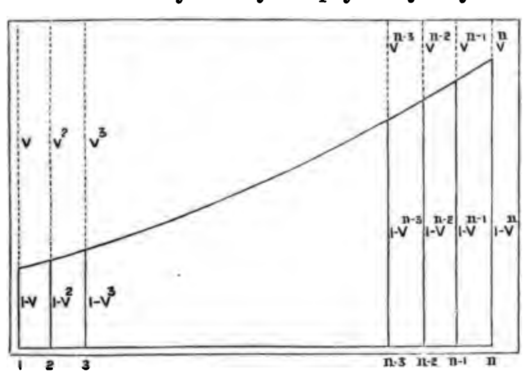
(25) Let us now consider the other case—that of a person advancing the sum $\frac{1-v^n}{i}$, in consideration of receiving at the end of each year for n years a payment of 1. At the end of the first year, the sum invested should yield a year's interest, that is, $i\times\frac{1-v^n}{i}=1-v^n$; and as the total amount then received is 1, the difference v^n must be considered as capital repaid. As the original capital was $v+v^2+\ldots+v^{n-1}+v^n$, and v^n is repaid at end of first year, the capital still outstanding will be

Each annual payment to be considered as partly repayment of capital and partly interest on capital outstanding.

$$v+v^2+\ldots+v^{n-1}=\frac{1-v^{n-1}}{i}.$$

At the end of second year, a year's interest will be due on $\frac{1-v^{n-1}}{i}$. This is equal to $i\times\frac{1-v^{n-1}}{i}=1-v^{n-1}$, leaving, as before, v^{n-1} out of the total amount 1 then received to be considered as capital repaid, and $\frac{1-v^{n-2}}{i}$ as the capital still outstanding. Similarly, at the end of the tth year, the portion of capital repaid out of the annual payment then made is v^{n-t+1}, the sum to be treated as interest for that year being $1-v^{n-t+1}$, and the capital still outstanding $\frac{1-v^{n-t}}{i}=a_{\overline{n-t}|}$. When $t=n$, the interest for the year is $1-v$, and the capital repaid out of the nth payment is v, the capital still outstanding being $\frac{1-v^0}{i}=0$.

The accompanying figure will serve to illustrate the formula for the value of an annuity for n years payable yearly:—



NOTE.—In this figure the lines denoting the tth payment are those corresponding to the abscissa, $n-t+1$.

Each payment of 1 is here partly denoted by ———— and partly by * * * *, the lines —— and * * * respectively representing the portions of each annual payment appropriated to repayment of capital, and interest on capital, respectively. It will be seen that the line joining the respective points of division forms a curve. This curve is what is known as the logarithmic curve, its property being that the ordinates corresponding to equidistant abscissæ are in geometric progression. Thus taking v^n, v^{n-1}, &c., corresponding to the abscissæ, n, $(n-1)$, &c., we see that they are in the constant ratio to one another of $\frac{1}{v}$ or $(1+i)$.

Redemption of capital by Accumulating Sinking Fund.

(26) It appears from the preceding demonstration that when the capital invested in the purchase of an annuity is $\frac{1-v^n}{i}$, the portion of capital repaid out of the first payment of the annuity is v^n. If, therefore, the capital invested in the purchase of an annuity be 1, the portion repaid out of the first payment of the corresponding annuity will be $\dfrac{v^n}{\frac{1-v^n}{i}} = \dfrac{1}{\frac{(1+i)^n-1}{i}}$, and the annual payment for n years which 1 will purchase will be $\dfrac{1}{\frac{(1+i)^n-1}{i}}$ + a year's interest on the original capital, *i.e.*, $\dfrac{1}{\frac{(1+i)^n-1}{i}}+i$.

It may be easily shown that this is equivalent to $\frac{1}{a_{\overline{n}|}}$, as it should be. For we have

$$\frac{1}{\frac{(1+i)^n-1}{i}}+i = \frac{i}{(1+i)^n-1}+i$$

$$= \frac{i(1+i)^n}{(1+i)^n-1}$$

$$= \frac{i}{1-(1+i)^{-n}}$$

$$= \frac{1}{\frac{1-v^n}{i}} = \frac{1}{a_{\overline{n}|}}.$$

Let $\dfrac{1}{\dfrac{(1+i)^n-1}{i}} = P_{\overline{n}|}$, then we have

$$1 = P_{\overline{n}|}\,\frac{(1+i)^n-1}{i}$$

$$= P_{\overline{n}|}\{1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}\}.$$

If, therefore, a capital of 1 be invested at a rate of interest i in the purchase of an annuity for n years, the amount of each annual payment will be $\dfrac{1}{a_{\overline{n}|}} = P_{\overline{n}|} + i$.

The portion of the capital of 1 repaid out of annuity received at end of the	first year	will be	$P_{\overline{n}\|}$
,, ,, ,,	second	,,	$P_{\overline{n}\|}(1+i)$
,, ,, ,,	third	,,	$P_{\overline{n}\|}(1+i)^2$
. . . . . .	. . .	. . .	. . .
,, ,, ,,	tth	,,	$P_{\overline{n}\|}(1+i)^{t-1}$
,, ,, ,,	nth	,,	$P_{\overline{n}\|}(1+i)^{n-1}$

As regards the borrower,—that is, the person who contracts to pay the annuity,—the capital of 1 may be said to be borrowed at the rate of interest i, to be repaid by means of an *accumulating sinking fund* of $P_{\overline{n}|}$ per annum.

Values of Annuities when the remunerative rate of interest on the Capital differs from the rate of interest at which the Sinking Fund will reproduce the Capital.

(27) Out of each annual payment of $P_{\overline{n}|}+i$ which a capital of 1 will purchase, it will be seen that we may consider i as constituting interest on capital, and $P_{\overline{n}|}$ as appropriated to repayment of the said capital, since the payments $P_{\overline{n}|}$ immediately invested at the same rate of interest i will have accumulated at the end of the n years to the capital 1. Suppose, however, that the person who proposes to invest his capital in the purchase of an annuity desires to provide against the contingency that he may not be able to re-invest the portions $P_{\overline{n}|}$ of each annual payment at the same rate of interest as that which he is receiving upon his capital: in other words, that he seeks to secure that the whole of his capital shall be invested for a time equal to the duration of the annuity at a given rate of interest. It is clear that the formula $a_{\overline{n}|} = \dfrac{1-v^n}{i}$

would not under these circumstances apply. If, however, the rate of interest desired to be received upon the capital be taken as i', and that at which re-investments can be made be taken to be i, so that $P_{\overline{n}|}$ will be determined from the formula

$$1=P_{\overline{n}|}\{1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}\}$$
$$=P_{\overline{n}|}\frac{(1+i)^n-1}{i};$$

then we shall have, denoting the value of an annuity of 1 by $a'_{\overline{n}|}$,

$$a'_{\overline{n}|}=\frac{1}{P_{\overline{n}|}+i'}.$$

The advantage of the general formula $a_{\overline{n}|}=\dfrac{1}{P_{\overline{n}|}+i}$ is that it brings into prominence two most important points:—

1. That each annual payment is made up of two portions, one portion consisting of interest upon capital, and the other forming a constant or uniform sinking fund to be re-invested so as at the end of the time to have accumulated to the original principal.
2. That the rate of interest at which the annual instalments of sinking fund are invested as received may or may not be the same as that which the capital bears for the time being.

Each annual payment to be considered as providing interest, at remunerative rate, on entire capital, and sinking fund instalment for accumulation at reproductive rate of interest.

We have now to consider what portion of the sum originally advanced under these conditions will be paid off at the end of each year. In the ordinary case, where the capital outstanding and the sinking fund for accumulation to replace the capital are bearing interest at the same rate, the matter is simple enough, as already shown; but the case under consideration has to be differently dealt with. Effect has to be given to the stipulation that the entire capital originally invested has to bear interest for the whole period of the annuity at one rate of interest, whereas the sinking fund for accumulation to replace the capital bears another rate of interest.

Now at the end of the first year $\dfrac{1}{P_{\overline{n}|}+i'}$ being the sum invested so as to bear interest for the entire n years at the rate i', the sinking

fund for accumulation is $1-\dfrac{i'}{P_{\overline{n}|}+i'}=\dfrac{P_{\overline{n}|}}{P_{\overline{n}|}+i'}$, and at the end of the first year the value of the $(n-1)$ future payments of 1 annually under the same conditions is $\dfrac{1}{P_{\overline{n-1}|}+i'}$, so that the amount of capital really paid off at the end of the first year is $\dfrac{1}{P_{\overline{n}|}+i'}-\dfrac{1}{P_{\overline{n-1}|}+i'}$ where $P_{\overline{n-1}|}\dfrac{(1+i)^{n-1}-1}{i}=1$. In other words, if at the beginning of the second year the borrower wished to cancel the contract, $\dfrac{1}{P_{\overline{n-1}|}+i'}$ is the sum down he should have to pay, as it would be considered that $\dfrac{1}{P_{\overline{n}|}+i'}-\dfrac{1}{P_{\overline{n-1}|}+i'}$ had been repaid at the end of the first year. As a matter of fact, at the end of the first year the lender receives a payment of 1 composed of $\dfrac{i'}{P_{\overline{n}|}+i'}$ on account of interest on his capital, and $\dfrac{P_{\overline{n}|}}{P_{\overline{n}|}+i'}$ as the instalment of sinking fund for accumulation, but we cannot consider the amount paid off as $\dfrac{P_{\overline{n}|}}{P_{\overline{n}|}+i'}$ because effect has to be given to the stipulation that interest at the rate i' has to be paid on the entire capital for the whole period of n years. We shall now proceed to show that if $\dfrac{1}{P_{\overline{n}|}+i'}-\dfrac{1}{P_{\overline{n-1}|}+i'}$ be denoted by V, then $\dfrac{P_{\overline{n}|}}{P_{\overline{n}|}+i'}$, the instalment of sinking fund for accumulation, is equivalent to

$$V\left\{1+(i'-i)\frac{1}{P_{\overline{n-1}|}+i}\right\};$$

that is, is equivalent to the amount actually repaid at the end of the first year, together with the present value of an annuity at the rate of interest i for the remaining $n-1$ years of the difference between the interest on the amount repaid, calculated at the remunerative rate i', and the reproductive rate i respectively.

We have $$V\left(1+\frac{i'-i}{P_{\overline{n-1}|}+i}\right)=V\frac{P_{\overline{n-1}|}+i'}{P_{\overline{n-1}|}+i}$$

But $$P_{\overline{n}|}\frac{(1+i)^n-1}{i}=1=P_{\overline{n-1}|}\frac{(1+i)^{n-1}-1}{i};$$

$$\therefore P_{\overline{n}|}\{1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}\}=P_{\overline{n-1}|}\{1+(1+i)+(1+i)^2+\ldots+(1+i$$

$$\therefore P_{\overline{n}|}\{(1+i)^{-(n-1)}+(1+i)^{-(n-2)}+\ldots+1\}=P_{\overline{n-1}|}\{(1+i)^{-(n-1)}+(1+i)^{-(n-2)}+\ldots(1+$$

$$\therefore\ P_{\overline{n}|}(1+a_{\overline{n-1}|})=P_{\overline{n-1}|}a_{\overline{n-1}|}$$

$$\therefore\ a_{\overline{n-1}|}=\frac{1}{P_{\overline{n-1}|}+i}=\frac{P_{\overline{n}|}}{P_{\overline{n-1}|}-P_{\overline{n}|}}.$$

Also, $$V=\frac{1}{P_{\overline{n}|}+i'}-\frac{1}{P_{\overline{n-1}|}+i'}=\frac{P_{\overline{n-1}|}-P_{\overline{n}|}}{(P_{\overline{n}|}+i')(P_{\overline{n-1}|}+i')}.$$

$$\therefore\ V\frac{P_{\overline{n-1}|}+i'}{P_{\overline{n-1}|}+i}=\frac{P_{\overline{n-1}|}-P_{\overline{n}|}}{(P_{\overline{n}|}+i')(P_{\overline{n-1}|}+i')}\cdot\frac{P_{\overline{n}|}}{P_{\overline{n-1}|}-P_{\overline{n}|}}\cdot(P_{\overline{n-1}|}+i')$$

$$=\frac{P_{\overline{n}|}}{P_{\overline{n}|}+i'}.$$

Corresponding results may be shown to hold at the end of any year previous to the termination of the annuity.

For instance, take the time when the tth payment of annuity has just been made, that is, at the end of t years. We have

$$P_{\overline{n}|}\{1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}=1$$

$$=P_{\overline{n-t}|}\{1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-t-1}\}$$

$$\therefore\ P_{\overline{n}|}\{v^{n-t}+v^{n-t-1}+\ldots+v+1+(1+i)+(1+i)^2+\ldots+(1+i)^{t-1}\}$$

$$=P_{\overline{n-t}|}\{v^{n-t}+v^{n-t-1}+\ldots+v\}$$

$$\therefore\ P_{\overline{n}|}\cdot\left\{a_{\overline{n-t}|}+\frac{(1+i)^t-1}{i}\right\}=P_{\overline{n-t}|}\cdot a_{\overline{n-t}|}$$

$$\therefore\ a_{\overline{n-t}|}=\frac{1}{P_{\overline{n-t}|}+i}=\frac{P_{\overline{n}|}\cdot\frac{(1+i)^t-1}{i}}{P_{\overline{n-t}|}-P_{\overline{n}|}}.$$

Denoting by ${}_tV$ the amount that may be considered to have been repaid by the end of t years, we have

$${}_tV=\frac{1}{P_{\overline{n}|}+i'}-\frac{1}{P_{\overline{n-t}|}+i'}$$

$$=\frac{P_{\overline{n-t}|}-P_{\overline{n}|}}{(P_{\overline{n}|}+i')(P_{\overline{n-t}|}+i')},$$

and we have $${}_tV\left(1+\frac{i'-i}{P_{\overline{n-t}|}+i}\right)={}_tV\cdot\frac{P_{\overline{n-t}|}+i'}{P_{\overline{n-t}|}+i}.$$

Substituting for $\frac{1}{P_{\overline{n-t}|}+i}$ its value obtained above,

viz.,
$$\frac{P_{\overline{n}|}\dfrac{(1+i)^t-1}{i}}{P_{\overline{n-t}|}-P_{\overline{n}|}}, \text{ we get ultimately}$$

$$_tV\left(1+\frac{i'-i}{P_{\overline{n-t}|}+i}\right)=\frac{P_{\overline{n-t}|}-P_{\overline{n}|}}{(P_{\overline{n}|}+i')(P_{\overline{n-t}|}+i')}\cdot\frac{(P_{\overline{n-t}|}+i')P_{\overline{n}|}\dfrac{(1+i)^t-1}{i}}{P_{\overline{n-t}|}-P_{\overline{n}|}}$$
$$=\frac{P_{\overline{n}|}}{P_{\overline{n}|}+i'}\cdot\frac{(1+i)^t-1}{i}.$$

If $i'-i=0$, that is, if there be only one rate of interest both for remuneration and re-investment, the amount to be considered as repaid by the end of t years will be exactly the amount accumulated in the sinking fund, that is,

$$_tV=\frac{P_{\overline{n}|}}{P_{\overline{n}|}+i}\frac{(1+i)^t-1}{i}$$
$$=\frac{P_{\overline{n-t}|}-P_{\overline{n}|}}{(P_{\overline{n}|}+i)\,(P_{\overline{n-t}|}+i)}$$
$$=\frac{(P_{\overline{n-t}|}-P_{\overline{n}|})\,a_{\overline{n-t}|}}{P_{\overline{n}|}+i}.$$

(28) Reverting now to the formulas of Articles 21 and 22, if $n=\infty$, the formula for the amount of an annuity becomes **Amounts and present values of perpetuities.**

$$\frac{(1+i)^\infty-1}{i},$$

that is, an infinitely large quantity.

The formula for the present value of the annuity becomes

$$a_\infty=\frac{1-v^\infty}{i};$$

and as v is always less than 1, v^∞ will become less than any finite quantity, and thus the value of a perpetual annuity will be $a_\infty=\frac{1}{i}$.

The present value of a perpetual annuity or a perpetuity may be otherwise obtained. For, using the method of reasoning of Art. 23, a capital of 1 is equivalent to the present value of an annuity of i, together with the present value of the said capital at the end of the term. The term being in this case of infinite duration, the present value of the capital is 0, and 1 therefore is equivalent in value to a perpetuity of i, so that the value of a perpetuity of 1 will be $\frac{1}{i}$.

Further consideration of the formulas for amounts and present values of annuities.

(29) Taking again the formulas for the amount and present value of an annuity for a term of n years, we have

$$\frac{(1+i)^n-1}{i}=(1+i)^n\times\frac{1}{i}-\frac{1}{i}$$

$$=(1+i)^n\{v+v^2+\ldots+v^n+v^{n+1}+\ldots ad\ inf.\}-\frac{1}{i}$$

$$=\{(1+i)^{n-1}+(1+i)^{n-2}+\ldots+(1+i)+1+v+v^2+\&c.\ ad\ inf$$

$$=\text{Amount of an annuity for } n \text{ years }+\frac{1}{i}-\frac{1}{i}$$

$$=\text{Amount of an annuity for } n \text{ years.}$$

Similarly, $\frac{1-v^n}{i}=\frac{1}{i}-v^n\times\frac{1}{i}$

$$=v+v^2+\ldots+v^n+v^{n+1}+\ldots ad\ inf.-v^n\times\frac{1}{i}$$

$$=v+v^2+\ldots+v^n+v^n(v+v^2+\ldots ad\ inf.)-v^n\times\frac{1}{i}$$

$$=\text{Present value of an annuity for } n \text{ years}+v^n\times\frac{1}{i}-v^n\times\frac{1}{i}$$

$$=\text{Present value of an annuity for } n \text{ years.}$$

Connection between annuities for n and $n+1$ years.

The connection between the amount of an annuity for n years and for $n+1$ years, is easily seen as follows:—

$$1+(1+i)+(1+i)^2+\ldots+(1+i)^n$$
$$=1+(1+i)\{1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}\},$$

or

Amount of annuity for $(n+1)$ years

$$=1+(1+i)\times\text{Amount of annuity for } n \text{ years.}$$

Similarly, for the connection between the value of an annuity for n years and for $n+1$ years, we have

$$v+v^2+\ldots+v^{n+1}=v\{1+v+v^2+\ldots v^n\},$$

or

$$a_{\overline{n+1}|}=v(1+a_{\overline{n}|}).$$

Formulas for annuities payable in advance.

Let us now suppose the annual payments, instead of being due at the end of successive years, to be due at the beginning of each year, then the amount of the annuity will evidently be the sum of the series

$$(1+i)+(1+i)^2+\ldots+(1+i)^n=1+(1+i)+(1+i)^2+\ldots+(1+i)^n-1,$$

or

$$\left.\begin{array}{c}\text{Amount of annuity in}\\ \text{advance for } n \text{ years}\end{array}\right\}=\left\{\begin{array}{c}\text{Amount of annuity for } (n+1)\\ \text{years} - 1.\end{array}\right.$$

Similarly, for the present value of the annuity in advance, we have

$$1+v+v^2+\ldots+v^{n-1}=1+a_{\overline{n-1|}},$$

or

$$\left.\begin{array}{c}\text{Present value of annuity}\\ \text{in advance for } n \text{ years}\end{array}\right\}=\left\{\begin{array}{c}\text{Present value of annuity for}\\ n-1 \text{ years} + 1.\end{array}\right.$$

As annuity tables are usually calculated on the assumption that the payments are due at the end of each year, the above formulas enable us at once to deduce the corresponding annuities when the payments are due at the beginning of each year.

Again, for the value of a perpetuity in advance, we have

$$\begin{aligned}\text{Value of perpetuity in advance} &= 1+v+v^2+\&\text{c.} \ldots \textit{ad inf.}\\ &= 1+\frac{1}{i}\\ &= \frac{1+i}{i}\\ &= \frac{1}{iv}=\frac{1}{1-v}=\frac{1}{d}. \quad \text{(Art. 14.)}\end{aligned}$$

Annuities whose term is not an exact number of years.

(30) We have hitherto only considered annuities where the payments are made at the end of each year, but it is possible that the circumstances may be such that the annuity is to run for a given number of years and a portion of a year, say, $\frac{1}{m}$th of a year.

As ordinarily calculated.

If we take the ordinary method of calculating such an annuity, there would be payable in the $(n+1)$th year, at the end of the interval $\frac{1}{m}$ of a year, the sum $\frac{1}{m}$, and the present value of such an annuity would be denoted by the sum of the series

$$v+v^2+\ldots+v^n+\frac{1}{m}v^{n+\frac{1}{m}}=a_{\overline{n|}}+\frac{1}{m}v^{n+\frac{1}{m}}.$$

If we employ the method of reasoning already made use of, that is, to consider 1 as equivalent to the value of an annuity of the interest

for the term, together with the present value of the capital at the end of the term, we shall have

$$1 = \text{Value of annuity of } i \text{ for } \left(n+\frac{1}{m}\right) \text{ years } + v^{n+\frac{1}{m}}.$$

Now the interest on 1 for each of the n whole years would be i, and for the $\frac{1}{m}$th part of the $(n+1)$th year, the interest accrued at the end of the $\frac{1}{m}$th part of the $(n+1)$th year would be $(1+i)^{\frac{1}{m}}-1$, on the assumption of compound interest (Art. 4): hence we shall have

$$i(v+v^2+v^3+\ldots+v^n)+\{(1+i)^{\frac{1}{m}}-1\}v^{n+\frac{1}{m}}+v^{n+\frac{1}{m}}=1,$$

or $$i\left\{v+v^2+v^3+\ldots+v^n+\frac{(1+i)^{\frac{1}{m}}-1}{i}v^{n+\frac{1}{m}}\right\}=1-v^{n+\frac{1}{m}}.$$

If we take $a_{\overline{n+\frac{1}{m}}|}$ to denote an annuity consisting of n annual payments of 1, and a further payment of $\frac{(1+i)^{\frac{1}{m}}-1}{i}$ at the end of the $\frac{1}{m}$th part of the $\overline{n+1}|$th year, we should have

$$ia_{\overline{n+\frac{1}{m}}|}=1-v^{n+\frac{1}{m}}$$

or $$a_{\overline{n+\frac{1}{m}}|}=\frac{1-v^{n+\frac{1}{m}}}{i}.$$

As already stated, the ordinary practice would be to pay $\frac{1}{m}$ at the end of the $\frac{1}{m}$th part of the $(n+1)$th year, *i.e.*, in proportion to the time elapsed, whereas, if the payment then made be $\frac{(1+i)^{\frac{1}{m}}-1}{i}$, it would bear the same proportion to a year's payment as the interest accrued at the end of $\frac{1}{m}$th of a year bears to a year's interest—that is, would be in proportion to the interest accrued.

Now $(1+i)^{\frac{1}{m}}-1<\frac{i}{m}$, and therefore $\frac{(1+i)^{\frac{1}{m}}-1}{i}<\frac{1}{m}$, so that the value of the annuity of practice would be greater than $\frac{1-v^{n+\frac{1}{m}}}{i}$. (See Art. (6).)

(31) If we have an annuity to run for n years, payment being deferred t years, then it is clear that its present value is denoted by the series **Deferred Annuities.**

$$v^{t+1}+v^{t+2}+\,..\,+v^{t+n}=v^t(v+v^2+\,..\,+v^n)$$
$$=v^t\frac{1-v^n}{i}$$
$$=v^t a_{\overline{n}|}.$$

And if the annuity is deferred $t+\frac{1}{q}$ years, the present value would be $v^{t+\frac{1}{q}}a_{\overline{n}|}$.

Similarly, the above deferred annuity may otherwise be considered as equivalent to an annuity for $n+t$ years less an annuity for t years, thus—

The present value of an annuity for n years deferred t years

$$=v^{t+1}+v^{t+2}+\,..\,+v^{t+n}$$
$$=(v+v^2+\,..\,+v^t+v^{t+1}+\,..\,+v^{t+n})-(v+v^2+\,..\,+v^t)$$
$$=\frac{1-v^{n+t}}{i}-\frac{1-v^t}{i}$$
$$=a_{\overline{n+t}|}-a_{\overline{t}|}.$$

If the annuity be a perpetuity deferred t years, its present value is clearly $v^t\times\frac{1}{i}=\frac{1}{i}-\frac{1-v^t}{i}=a_\infty-a_{\overline{t}|}$.

(32) As a numerical illustration of the preceding Articles of this Chapter, let us take in the first instance an annuity for 20 years, interest at the rate of 5 per-cent. So that $n=20$, $i=\cdot05$. **Numerical illustrations of preceding articles.**

(*a*) Let us first find the amount and present value of the annuity from the formula (Art. 21 and 22). **Articles 21 and 22.**

Amount of annuity$=\dfrac{(1\cdot05)^{20}-1}{\cdot05}$. Present value$=\dfrac{1-(1\cdot05)^{-20}}{\cdot05}$.

We have	$\log 1\cdot05 = 0\cdot021189$	$\therefore\ \log(1\cdot05)^{-1}=\bar{1}\cdot978811$
	$20\log 1\cdot05 = 0\cdot42378$	$\log(1\cdot05)^{-20}=\bar{1}\cdot57622$
	$(1\cdot05)^{20}= 2\cdot6533$	$(1\cdot05)^{-20}= \cdot37689$
	$(1\cdot05)^{20}-1= 1\cdot6533$	$1-(1\cdot05)^{-20}= \cdot62311$
	$\dfrac{(1\cdot05)^{20}-1}{\cdot05}=33\cdot066$	$\dfrac{1-(1\cdot05)^{-20}}{\cdot05}= 12\cdot4622$

Again, present value of annuity $=(1\cdot05)^{-20}\times$ amount of annuity,

$$\log 33\cdot066 = 1\cdot51938$$
$$\log(1\cdot05)^{-20} = \bar{1}\cdot57622$$
$$\overline{1\cdot09560} = \log^{-1} 12\cdot462.$$

Articles 24-26. (*b*) Let us now consider the purchase of such an annuity from the point of view both of the person who buys it and of the person who grants it. The former we will call the lender, the latter the borrower.

The lender advances the sum of 12·4622 in consideration of receiving from the borrower a payment of 1 at the end of every year for 20 years.

The borrower, in consideration of an advance of this sum of 12·4622, undertakes to pay 1 at the end of every year for 20 years.

Now, the borrower considers the payment of 1 made

At end of year	1	to consist of	·6231	interest at 5 per cent	on 12·4622	and ·3769	in repayment of advance	leaving 12·0853	still unpaid	$=a$
„	2	„	·6043	„	12·0853	„ ·3957	„	„ 11·6896	„	$=a$
„	3	„	·5845	„	11·6896	„ ·4155	„	„ 11·2741	„	$=a$
„	4	„	·5637	„	11·2741	„ ·4363	„	„ 10·8378	„	$=a$
„	5	„	·5419	„	10·8378	„ ·4581	„	„ 10·3797	„	$=a$
„	6	„	·5190	„	10·3797	„ ·4810	„	„ 9·8987	„	$=a$
„	7	„	·4949	„	9·8987	„ ·5051	„	„ 9·3936	„	$=a$
„	8	„	·4697	„	9·3936	„ ·5303	„	„ 8·8633	„	$=a$
„	9	„	·4432	„	8·8633	„ ·5568	„	„ 8·3065	„	$=a$
„	10	„	·4153	„	8·3065	„ ·5847	„	„ 7·7218	„	$=a$
„	11	„	·3861	„	7·7218	„ ·6139	„	„ 7·1079	„	$=a$
„	12	„	·3554	„	7·1079	„ ·6446	„	„ 6·4633	„	$=a$
„	13	„	·3232	„	6·4633	„ ·6768	„	„ 5·7865	„	$=a$
„	14	„	·2893	„	5·7865	„ ·7107	„	„ 5·0758	„	$=a$
„	15	„	·2538	„	5·0758	„ ·7462	„	„ 4·3296	„	$=a$
„	16	„	·2165	„	4·3296	„ ·7835	„	„ 3·5461	„	$=a$
„	17	„	·1773	„	3·5461	„ ·8227	„	„ 2·7234	„	$=a$
„	18	„	·1362	„	2·7234	„ ·8638	„	„ 1·8596	„	$=a$
„	19	„	·0930	„	1·8596	„ ·9070	„	„ ·9526	„	$=a$
„	20	„	·0476	„	·9526	„ ·9524	„	„ *Nil*	„	
						12·4620				

NOTE.—The exact total should be 12·4622. The difference arises from the fact that only 4 places of decimals have been used.

As regards the lender, matters would stand thus:—He will receive 1 at the end of every year to be thus disposed of: ·6231 interest at 5 per-cent on his capital of 12·4622 and ·3769 to be forthwith invested at the same rate of interest to be accumulated for the purpose of replacing the said capital by the time the annuity ceases. In this way the amount accumulated towards replacement of capital will be as follows:—

At the end of year	1	the amount accumulated towards replacement of capital will be	·3769	= ·3769
,,	2	,, ,,	·3769 + 1·05 × ·3769	= ·7726
,,	3	,, ,,	·3769 + 1·05 × ·7726	= 1·1881
,,	4	,, ,,	·3769 + 1·05 × 1·1881	= 1·6244
,,	5	,, ,,	·3769 + 1·05 × 1·6244	= 2·0825
,,	6	,, ,,	·3769 + 1·05 × 2·0825	= 2·5635
,,	7	,, ,,	·3769 + 1·05 × 2·5635	= 3·0686
,,	8	,, ,,	·3769 + 1·05 × 3·0686	= 3·5989
,,	9	,, ,,	·3769 + 1·05 × 3·5989	= 4·1557
,,	10	,, ,,	·3769 + 1·05 × 4·1557	= 4·7404
,,	11	,, ,,	·3769 + 1·05 × 4·7404	= 5·3543
,,	12	,, ,,	·3769 + 1·05 × 5·3543	= 5·9989
,,	13	,, ,,	·3769 + 1·05 × 5·9989	= 6·6757
,,	14	,, ,,	·3769 + 1·05 × 6·6757	= 7·3864
,,	15	,, ,,	·3769 + 1·05 × 7·3864	= 8·1326
,,	16	,, ,,	·3769 + 1·05 × 8·1326	= 8·9161
,,	17	,, ,,	·3769 + 1·05 × 8·9161	= 9·7388
,,	18	,, ,,	·3769 + 1·05 × 9·7388	= 10·6026
,,	19	,, ,,	·3769 + 1·05 × 10·6026	= 11·5096
,,	20	,, ,,	·3769 + 1·05 × 11·5096	= 12·4622

the amount annually re-invested ·3769 is called the Sinking Fund for accumulation.

Article 27. (c) Let us now consider the case where the borrower, as before, pays 5 per-cent interest upon the capital, but it is necessary that allowance be made for the fact that the lender can only reinvest his money at 4 per-cent. In this case, it is clear that the amount of the advance must be reduced, or the annual payments by the borrower, extending over 20 years, increased. If we take the former case, then by Article 26 the amount of the advance will be given by the formula $a'_{\overline{20}|} = \dfrac{1}{P_{\overline{20}|} + \cdot 05}$.

We have first to determine $P_{\overline{20}|}$ from the formula $P_{\overline{20}|} = \dfrac{1}{\dfrac{(1\cdot 04)^{20} - 1}{\cdot 04}}$.

$$\log 1\cdot 04 = \cdot 017033$$

$$20 \log 1\cdot 04 = \cdot 34066$$

$$(1\cdot 04)^{20} = 2\cdot 1911$$

$$(1\cdot 04)^{20} - 1 = 1\cdot 1911$$

$$\frac{(1\cdot 04)^{20} - 1}{\cdot 04} = 29\cdot 7775 \text{ and } \log P_{\overline{20}|} = -\log 29\cdot 7775 = -1\cdot 473888$$

$$= \bar{2}\cdot 526112;$$

$$\therefore\ P_{\overline{20}|} = \cdot 033583$$

$$\text{Hence } \log a'_{\overline{20}|} = \log \frac{1}{P_{\overline{20}|} + \cdot 05} = -\log\ \cdot 083583$$

$$= -\ \bar{2}\cdot 922118$$

$$= -(-1\cdot 077882)$$

$$= 1\cdot 077882$$

$$\therefore\ a'_{\overline{20}|} = 11\cdot 9642$$

As shown above, if the value of the annuity had been calculated in the ordinary way at 5 per-cent interest, its value would be $a_{\overline{20}|} = 12\cdot 4622$.

The following table will show the state of affairs as between lender and borrower at the end of each year.

Year. (t)	Interest received at 5 per-cent.		Sinking Fund for accumulation at 4 per-cent interest.		$\frac{1}{P_{\overline{20-t}\rvert} + \cdot 05}$ = Redemption Money.	$P_{\overline{20}\rvert} \frac{(1\cdot 04)^t - 1}{\cdot 04}$ = Accumulated Amount of Sinking Fund.
1	·59821 (being 5 per-cent on 11·9642)	+	·40179 (being ·033583 × 11·9642)	= 1	11·6092	·40179
2	,,		,,	,,	11·2366	·81965
3	,,		,,	,,	10·8462	1·2542
4	,,		,,	,,	10·4362	1·7062
5	,,		,,	,,	10·0059	2·1762
6	,,		,,	,,	9·55201	2·6651
7	,,		,,	,,	9·07902	3·1734
8	,,		,,	,,	8·57986	3·7022
9	,,		,,	,,	8·05483	4·2522
10	,,		,,	,,	7·50238	4·8239
11	,,		,,	,,	6·92072	5·4185
12	,,		,,	,,	6·30813	6·0373
13	,,		,,	,,	5·66220	6·6805
14	,,		,,	,,	4·98103	7·3495
15	,,		,,	,,	4·26208	8·0454
16	,,		,,	,,	3·50274	8·7692
17	,,		,,	,,	2·70015	9·5214
18	,,		,,	,,	1·85118	10·3042
19	,,		,,	,,	·952382	11·1180
20	,,		,,	,,	*Nil*	11·9642

Thus, at the end of the first year, the lender receives 1, consisting of ·59821 interest at 5 per-cent on his capital of 11·9642, and ·40179 as sinking fund, to be accumulated at 4 per-cent interest, so as to reproduce his capital of 11·9642 at the end of the 20 years. The amount of the said capital which may be considered as still outstanding is $\frac{1}{P_{\overline{19}\rvert} + \cdot 05} = 11\cdot 6092$, so that at the end of the year the borrower may be considered to have paid off $11\cdot 9643 - 11\cdot 6092 = \cdot 3551$; and it will be found that

$$\cdot 3551 \left\{ 1 + \frac{\cdot 01}{P_{\overline{19}\rvert} + \cdot 04} \right\} = \cdot 40179.$$

In other words, that the amount then in the sinking fund is equivalent to the amount that may be considered as actually

repaid, + value of an annuity of the difference between 4 per-cent and 5 per-cent interest on the said ·3551 for 19 years, computed at the reproductive rate of 4 per-cent.

$$\frac{1}{P_{\overline{19}|}+\cdot 04}=\frac{1-(1\cdot 04)^{-19}}{\cdot 04}=13\cdot 1334 \text{ (for method of calculation, see above.)}$$

$$1+\cdot 01\times\frac{1}{P_{\overline{19}}+\cdot 04}=1\cdot 131334=\log^{-1}0\cdot 053591$$

$$\log\cdot 3551=\bar{1}\cdot 550351$$

$$\bar{1}\cdot 603942=\log\cdot 40174$$

which is correct to 4 places of decimals.

Similarly it may be shown, that at the end of any other year the like result obtains.

ANNUITY PAYABLE AND INTEREST CONVERTIBLE AT THE SAME PERIODS, THE PERIOD BEING LESS THAN A YEAR.

(33) We have hitherto assumed the payment of the annuity to be made yearly, and the interest to be convertible yearly; and it is now proposed to consider the case where the annuity is payable, and the interest convertible, at periods of less than a year.

Formulas for amounts and present values of annuities.

Let us call that period $\frac{1}{m}$th of a year, and let the nominal rate of interest be assumed to be x. Then, if we denote the corresponding effective rate of interest by i, we have

$$\left(1+\frac{x}{m}\right)^m=1+i=\frac{1}{v}$$

$$\therefore\quad 1+\frac{x}{m}=(1+i)^{\frac{1}{m}}=v^{-\frac{1}{m}}.$$

Thus the amount of an annuity of 1 for n years, payable by instalments of $\frac{1}{m}$ at the end of each interval of $\frac{1}{m}$, will be denoted by the series

$$\frac{1}{m}\left\{1+\left(1+\frac{x}{m}\right)+\left(1+\frac{x}{m}\right)^2+\ldots+\left(1+\frac{x}{m}\right)^{mn-1}\right\}$$

$$=\frac{1}{m}\left\{1+(1+i)^{\frac{1}{m}}+(1+i)^{\frac{2}{m}}+\ldots+(1+i)^{n-\frac{1}{m}}\right\}$$

$$=\frac{1}{m}\frac{(1+i)^n-1}{(1+i)^{\frac{1}{m}}-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, substituting for $(1+i)$ its value $\left(1+\frac{x}{m}\right)^m$,

$$=\frac{1}{m}\frac{\left(1+\frac{x}{m}\right)^{mn}-1}{\left(1+\frac{x}{m}\right)-1}=\frac{\left(1+\frac{x}{m}\right)^{mn}-1}{x} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, the present value of the same annuity,

$$=\frac{1}{m}\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{m}}-1} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$=\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{x} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Other methods of obtaining the formulas for amounts and present values of annuities.

The formulas (1) and (2) may be otherwise obtained as follows:—

Principal of 1 + amount of annuity of x payable by instalments of $\frac{x}{m}$ at the end of each interval of $\frac{1}{m}$th of a year

= Amount of 1 at the end of n years when interest at the nominal rate of x is convertible m times a year

$$=\left(1+\frac{x}{m}\right)^{mn}$$

∴ Amount of an annuity of x payable by instalments of $\frac{x}{m}$ at the end of each interval of $\frac{1}{m}$th of a year $\Bigg\} = \left(1+\frac{x}{m}\right)^{mn}-1$

∴ Amount of an annuity of 1 payable by instalments of $\frac{1}{m}$ at the end of each interval of $\frac{1}{m}$th of a year $\Bigg\} = \dfrac{\left(1+\frac{x}{m}\right)^{mn}-1}{x}$

Otherwise thus:—As the interest yielded by a capital of 1 at the end of $\frac{1}{m}$th of a year, the effective rate of interest being i, is $(1+i)^{\frac{1}{m}}-1$, a capital of 1 will yield a payment of $(1+i)^{\frac{1}{m}}-1$ at the end of each interval of $\frac{1}{m}$th of a year, and the sum paid in the course of a year would be $m\{(1+i)^{\frac{1}{m}}-1\}$, so that

$$\begin{aligned}\text{Capital of } 1+\text{Amount of an annuity of } m\{(1+i)^{\frac{1}{m}}-1\}\\ &=\text{Amount of 1 at the end of } n \text{ years}\\ &=(1+i)^n.\end{aligned}$$

$$\therefore \left.\begin{array}{l}\text{Amount of an annuity of } m\{(1+i)^{\frac{1}{m}}-1\}\\ \text{for } n \text{ years payable by instalments of}\\ \{(1+i)^{\frac{1}{m}}-1\} \text{ at the end of each interval}\\ \text{of } \frac{1}{m}\text{th of a year}\end{array}\right\}=(1+i)^n-1$$

$$\therefore \left.\begin{array}{l}\text{Amount of an annuity of 1 for } n \text{ years}\\ \text{payable by instalments of } \frac{1}{m} \text{ at the end}\\ \text{of each interval of } \frac{1}{m}\text{th of a year}\end{array}\right\}=\frac{1}{m}\frac{(1+i)^n-1}{(1+i)^{\frac{1}{m}}-1}$$

Similarly, if we denote the present value of an annuity of 1 for n years payable by instalments of $\frac{1}{m}$ at the end of each interval of $\frac{1}{m}$th of a year by $a_{\overline{n}|}^{(m)}$, we have

Capital of 1 = Present value of 1 receivable n years hence,

+ Present value of annuity of the interest yielded by capital of 1 for n years.

or, $$(1+i)^{-n}+m\{(1+i)^{\frac{1}{m}}-1\}a_{\overline{n}|}^{(m)}=1;$$

$$\therefore \quad a_{\overline{n}|}^{(m)}=\frac{1}{m}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{m}}-1}$$

$$=\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{x}.$$

The formula for the present value of the annuity could be deduced from the formula for the amount of an annuity from the consideration (Art. 22) that it should be such a sum as, accumulated at interest at the nominal rate x, interest being convertible m times a year,—*i.e.*, at an effective rate i,—would amount to the same sum as the amount of an annuity under the same conditions. That is, denoting the present value of the annuity by $a_{\overline{n}|}^{(m)}$,

$$\left(1+\frac{x}{m}\right)^{mn} a_{\overline{n}|}^{(m)} = \frac{\left(1+\frac{x}{m}\right)^{mn}-1}{x}$$

$$\therefore\quad a_{\overline{n}|}^{(m)} = \frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{x}.$$

But $\left(1+\frac{x}{m}\right)^{m}=1+i$, and $x=m\{(1+i)^{\frac{1}{m}}-1\}$

$$\therefore\quad a_{\overline{n}|}^{(m)} = \frac{1}{m}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{m}}-1}.$$

(34) If we now proceed to analyze the respective payments of the annuity, we see that the interest on the capital invested is at the end of the first period of $\frac{1}{m}$,

Each payment to be considered as partly repayment of capital and partly interest on capital outstanding.

$$\{(1+i)^{\frac{1}{m}}-1\}a_{\overline{n}|}^{(m)} \quad\text{or}\quad \frac{x}{m}\cdot a_{\overline{n}|}^{(m)};$$

and as the first instalment of the annuity is $\frac{1}{m}$, the amount of capital repaid out of such instalment is $\frac{1}{m}$ (less interest on $a_{\overline{n}|}^{(m)}$ for $\frac{1}{m}$th of a year)

$$=\frac{1}{m}-\{(1+i)^{\frac{1}{m}}-1\}a_{\overline{n}|}^{(m)} \quad\text{or}\quad \frac{1}{m}-\frac{x}{m}\cdot a_{\overline{n}|}^{(m)}$$

$$=\frac{1}{m}-\frac{1-(1+i)^{-n}}{m} \quad\text{or}\quad \frac{1}{m}-\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{m}$$

$$=\frac{(1+i)^{-n}}{m} \quad\text{or}\quad \frac{\left(1+\frac{x}{m}\right)^{-mn}}{m}.$$

That is, just as in the case of the annuity payable yearly (Art. 25), the capital redeemed out of the first payment is the last term of the series $v+v^2+\ldots+v^n$, whose sum constitutes the original capital, so in this case the capital redeemed out of the first instalment of the annuity is the last term of the series $\frac{1}{m}\left\{v^{\frac{1}{m}}+v^{\frac{2}{m}}+\ldots+v^{n-\frac{1}{m}}+v^n\right\}$ whose sum constitutes the original capital.

Similarly, there is redeemed out of the second instalment, $\frac{v^{n-\frac{1}{m}}}{m}$,

" " third " $\frac{v^{n-\frac{2}{m}}}{m}$,

and so on.

Portion of capital redeemed by the payments made in the course of each year.

The total amount of capital redeemed the first year

$$=\frac{1}{m}\left\{v^n+v^{n-\frac{1}{m}}+v^{n-\frac{2}{m}}+\ldots+v^{n-\frac{m-1}{m}}\right\},$$

the total amount of capital redeemed the second year,

$$=\frac{1}{m}\left\{v^{n-1}+v^{n-1-\frac{1}{m}}+v^{n-1-\frac{2}{m}}+\ldots+v^{n-1-\frac{m-1}{m}}\right\},$$

and so on.

The expression for the total amount of capital redeemed the first year may be put into a more convenient form. For

$$\frac{1}{m}\left\{v^n+v^{n-\frac{1}{m}}+v^{n-\frac{2}{m}}+\ldots+v^{n-\frac{m-1}{m}}\right\}=\frac{v^n}{m}\left\{1+v^{-\frac{1}{m}}+v^{-\frac{2}{m}}+\ldots+v^{-\frac{m-1}{m}}\right\}$$

$$=\frac{v^n}{m}\left\{1+\left(1+\frac{x}{m}\right)+\left(1+\frac{x}{m}\right)^2+\ldots+\left(1+\frac{x}{m}\right)^{m-1}\right\}$$

$$=\frac{v^n}{m}\left\{\frac{\left(1+\frac{x}{m}\right)^m-1}{\frac{x}{m}}\right\}=v^n\frac{\overline{1+i}-1}{x}=v^n\times\frac{i}{x}.$$

Similarly, the amount redeemed in the course of the second year

$$=\frac{v^{n-1}}{m}\left\{\frac{\left(1+\frac{x}{m}\right)^m-1}{\frac{x}{m}}\right\}=v^{n-1}\times\frac{i}{x};$$

the amount redeemed in the course of the third year

$$=\frac{v^{n-2}}{m}\left\{\frac{\left(1+\frac{x}{m}\right)^m-1}{\frac{x}{m}}\right\}=v^{n-2}\times\frac{i}{x};$$

&c., &c.

the amount redeemed in the course of the nth year

$$=\frac{v}{m}\left\{\frac{\left(1+\frac{x}{m}\right)^m-1}{\frac{x}{m}}\right\}=v\times\frac{i}{x}.$$

Summing, we get the total amount redeemed, being the original capital

$$=\frac{i}{x}\left\{v+v^2+\ldots+v^n\right\};$$

that is,
$$\overset{(m)}{a}_{\overline{n}|}=\frac{i}{x}\cdot\frac{1-v^n}{i}=\frac{i}{x}a_{\overline{n}|}$$
$$=\frac{1-v^n}{x},$$

which is the formula previously obtained for $\overset{(m)}{a}_{\overline{n}|}$ (Art. 33).

Connection between annuity values at nominal and effective rates of interest.

(35) If now we have two annuities to run for n years, the one payable yearly and the other payable by instalments of $\frac{1}{m}$ at the end of each interval of $\frac{1}{m}$th of a year; and if the effective rate of interest is in both cases i, and in the latter case x the nominal rate convertible m times a year, then we know, from what has been previously shown, that

$$1=v^n+ia_{\overline{n}|}=v^n+x\overset{(m)}{a}_{\overline{n}|};$$

$$\therefore\quad \overset{(m)}{a}_{\overline{n}|}=\frac{i}{x}a_{\overline{n}|},\text{ as shown above.}$$

When, therefore, a table of values of $a_{\overline{n}|}$ at the rate i has been calculated, the values of annuities payable by instalments of $\frac{1}{m}$ at the end of each interval of $\frac{1}{m}$th of a year, the nominal rate of

interest x convertible m times a year being equivalent to an effective rate i, are found at once from the formula

$$a_{\overline{n}|}^{(m)} = \frac{i}{x} a_{\overline{n}|}.$$

As will be seen, however, from Table II. of Chapter I., the values of x corresponding to the values of i for which annuity values are commonly tabulated would be rarely, if ever, employed; and in practice the values of x usually employed (see Table I. of Chapter I.), are found to be such as have no corresponding values of i for which annuity tables have been constructed. Thus, for instance, given annuity tables calculated at the rate of interest $4\frac{1}{2}$ per-cent, interest convertible and annuity payable yearly, then by simple multiplication by the ratios

$$\frac{4{\cdot}5}{4{\cdot}450483}, \quad \frac{4{\cdot}5}{4{\cdot}425996}, \quad \frac{4{\cdot}5}{4{\cdot}409768}, \quad \frac{4{\cdot}5}{4{\cdot}401954}$$

respectively, we should at once obtain amounts and present values of annuities, having (see Table I., Chapter I.)—

(1) Interest convertible, and annuity payable half-yearly, at nominal rate of interest 4·450483 per-cent.

(2) Interest convertible, and annuity payable quarterly, at nominal rate of interest 4·425996 per-cent.

(3) Interest convertible, and annuity payable monthly, at nominal rate of interest 4·409768 per-cent.

(4) Interest convertible, and annuity payable daily, at nominal rate of interest 4·401954 per-cent.

Putting the formula for $a_{\overline{n}|}^{(m)}$ into the form

$$a_{\overline{n}|}^{(m)} = \frac{1}{m} \cdot \frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{\frac{x}{m}},$$

we see that the value of an annuity of 1 for n years payable by instalments of $\frac{1}{m}$ at the end of each interval of $\frac{1}{m}$th of a year, at the nominal rate of interest x, being convertible m times a year, is equal in value to an annuity of $\frac{1}{m}$ for mn years, interest at the

rate $\frac{x}{m}$ (Art. 11). Thus, for instance, an annuity of 1 for 20 years, interest at the nominal rate of 4 per-cent, and the annuity payable and interest convertible half-yearly, is equal in value to an annuity of $\frac{1}{2}$ for 40 years, interest at the rate of 2 per-cent. In this manner, a table of values of annuities payable yearly at a given rate of interest may be made use of to obtain the values of annuities payable at periods of less than a year, at another nominal rate of interest convertible at the same periods as the annuity is payable; but it will be seen that the method is necessarily only capable of limited application.

(For further illustration of this Article, see Example 5 of this Chapter, in Chapter V.)

Continuous annuities. Formulas for their amounts and present values.

(36) We have seen that if the annuity is payable by instalments at intervals of $\frac{1}{m}$th of a year, the nominal rate of interest x being convertible at the same periods, then

$$\text{Amount of an annuity for } n \text{ years} = \frac{\left(1+\frac{x}{m}\right)^{mn}-1}{x},$$

$$\text{Present value of do.} = \frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{x}.$$

Let us now assume m to be infinitely large, so that the intervals for payment of annuity and for conversion of interest are infinitely small, then the annuity is called a continuous annuity, and we have (Arts. 9-11)

$$\text{Amount of continuous annuity for } n \text{ years} = \frac{\epsilon^{nx}-1}{x},$$

$$\text{Present value of do.} = \frac{1-\epsilon^{-nx}}{x}.$$

When the limiting value of $\left(1+\frac{x}{m}\right)^m_{m=\infty}$ is taken to be equal to $1+i$, it is usual to put δ for the corresponding value of x, and then the above formulas would become $\frac{\epsilon^{n\delta}-1}{\delta}$ and $\frac{1-\epsilon^{-n\delta}}{\delta}$ respectively. This last value is denoted by $\bar{a}_{\overline{n}|}$, so that $\bar{a}_{\overline{n}|} = \frac{1-\epsilon^{-n\delta}}{\delta}$.

On a method for facilitating the calculation of continuous annuities.

A property should be noted here with regard to continuous annuities, which may be made use of in the determination of their amounts and present values.

We have $$\frac{\epsilon^{nx}-1}{x}=\frac{\epsilon^{nx}-1}{nx}\times n,$$

and $$\frac{1-\epsilon^{-nx}}{x}=\frac{\epsilon^{nx}-1}{nx}\times n\times\epsilon^{-nx}.$$

Hence $$\log\frac{\epsilon^{nx}-1}{x}=\log\frac{\epsilon^{nx}-1}{nx}+\log n,$$

$$\log\frac{1-\epsilon^{-nx}}{x}=\log\frac{\epsilon^{nx}-1}{nx}+\log n-nx\log\epsilon.$$

It follows, therefore, that if a table of values be given for the function $\log\frac{\epsilon^{nx}-1}{nx}$, corresponding to given values of nx, the calculation of amounts and present values of continuous annuities certain may be greatly facilitated.

Deferred annuities.

(37) From what has been shown in the case of annuities payable yearly and the interest convertible yearly (Art. 31), it is easily seen that the value of an annuity for n years deferred t years is

$$\left(1+\frac{x}{m}\right)^{-mt}\times\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{x}=\left(1+\frac{x}{m}\right)^{-mt}\times\frac{1}{m}\cdot\frac{1-v^n}{(1+i)^{\frac{1}{m}}-1},$$
$$=v^t\times a_{\overline{n}|}^{(m)}$$
$$=a_{\overline{n+t}|}^{(m)}-a_{\overline{t}|}^{(m)}.$$

If the annuities are continuous, then the value of a continuous annuity for n years to be entered upon at the end of t years, the effective rate of interest being i, is

$$v^t\bar{a}_{\overline{n}|}=\bar{a}_{\overline{n+t}|}-\bar{a}_{\overline{t}|}.$$

Perpetuities.

(38) If, in the formulas (3) and (4) of Article (33), we assume n, the number of years for which the annuity is to run, to be infinite, then, as in (Art. 28), $(1+i)^{-n}=\left(1+\frac{x}{m}\right)^{-mn}$ becomes a vanishing quantity, and we have for the present value of a perpetuity

$$a_{\infty}^{(m)}=\frac{1}{m}\frac{1}{(1+i)^{\frac{1}{m}}-1}=\frac{1}{x}.$$

From this it follows that if x be the nominal rate of interest, then the value of a perpetuity is the same, no matter how often interest is convertible in the course of a year, provided the payments of the perpetuity are made as often as interest is convertible.

Thus, if 5 per-cent be the nominal rate of interest, the value of a perpetuity will be 20 years' purchase, no matter whether interest is convertible and annuity payable yearly, interest convertible and annuity payable half-yearly, quarterly, or any other interval, provided always that the annuity is payable at the same periods as the interest is convertible. From general considerations, it will appear that the oftener interest is convertible, the greater the effective rate of interest, and consequently the less the value of the perpetuity; but it has also to be borne in mind that the oftener the payments of annuity are made, the greater would appear to be the value of the perpetuity. As a matter of fact, the two conditions exactly counterbalance one another. This may be shown algebraically, thus:

Let x be the nominal rate of interest, and let interest be convertible and annuity payable at every $\frac{1}{m}$th period of a year. Then the payments made in the course of any year will have accumulated at the end of the year to

$$\frac{1}{m}\left\{1+\left(1+\frac{x}{m}\right)+\left(1+\frac{x}{m}\right)^2+\left(1+\frac{x}{m}\right)^3 +\&c.+\left(1+\frac{x}{m}\right)^{m-1}\right\}=\frac{1}{m}\cdot\frac{\left(1+\frac{x}{m}\right)^m-1}{\frac{x}{m}}.$$

Let, as usual, i be the corresponding effective rate of interest, so that $\left(1+\frac{x}{m}\right)^m=1+i$, and it will follow that the payments made in the course of a year will have amounted at the end of that year to $\frac{1}{m}\cdot\frac{1+i-1}{\frac{x}{m}}=\frac{i}{x}$. Accordingly the value of the perpetuity will be the same as if we consider a payment of $\frac{i}{x}$, instead of 1, made at the end of each year, and interest convertible

yearly at the rate i. Consequently, the value of the perpetuity,

$$i.e., \; a_{\infty}^{(m)} = \frac{1}{i} \times \frac{i}{x}$$

$$= \frac{1}{x}.$$

This result would also follow directly from the formula obtained in Article 35. The formula is $a_{\overline{n}|}^{(m)} = \frac{i}{x} a_{\overline{n}|}$.

Making $n = \infty$, we have $\quad a_{\infty}^{(m)} = \frac{i}{x} a_{\infty} = \frac{1}{x}$.

(A further illustration will be found in Example 7 of this Chapter, in Chapter V.)

ANNUITY PAYABLE AND INTEREST CONVERTIBLE AT PERIODS OF DIFFERENT DURATION.

Although generally in practice it is not usual to make any difference between the periods when instalments of an annuity are payable and when interest is assumed to be convertible, yet theoretically there is no necessity for the two periods to coincide. For instance, the instalments of an annuity may be payable quarterly, and interest convertible half-yearly, or *vice versâ*. It is proposed briefly to consider such cases.

(39) *Annuity payable yearly and Interest convertible* m *times a year*:—

If x be the nominal rate of interest convertible m times a year, the series whose sums denote the amount and present value of an annuity of 1 for n years are respectively

Amount of annuity of 1 for n years

$$= 1 + \left(1 + \frac{x}{m}\right)^{m} + \left(1 + \frac{x}{m}\right)^{2m} + \ldots + \left(1 + \frac{x}{m}\right)^{m(n-1)},$$

and present value of annuity of 1 for n years

$$=\left(1+\frac{x}{m}\right)^{-m}+\left(1+\frac{x}{m}\right)^{-2m}+\ldots+\left(1+\frac{x}{m}\right)^{-mn}.$$

The sums of these series are respectively

$$\frac{\left(1+\frac{x}{m}\right)^{mn}-1}{\left(1+\frac{x}{m}\right)^{m}-1} \text{ and } \frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{\left(1+\frac{x}{m}\right)^{m}-1}.$$

If we assume i to be the effective rate of interest, so that $\left(1+\frac{x}{m}\right)^{m}=1+i$, we have

$$\text{Amount of annuity for } n \text{ years} = \frac{(1+i)^n-1}{i},$$

$$\text{and the present value of do.} = \frac{1-(1+i)^{-n}}{i}.$$

These formulas are the same as have already been obtained for an annuity payable yearly, the rate of interest being i convertible yearly (see Arts. 21, 22). It follows, therefore, that the amount and present value of an annuity payable yearly, interest at a given nominal rate x convertible m times a year, are equal to the corresponding amount and present value for an annuity payable yearly at the corresponding effective rate of interest i. Thus the amount and present value of an annuity payable yearly, with interest at the nominal rate $\delta=\log_\epsilon(1+i)$ convertible momently, are the same as for an annuity payable yearly with interest convertible yearly at the rate i.

Similar remarks apply to deferred annuities and perpetuities.

(40) *Annuity payable* m *times a year, and Interest convertible yearly* :—

If n be the number of years the annuity has to run, i the rate of interest convertible yearly, then evidently

$$\text{Amount of annuity} = \frac{1}{m}\left\{1+(1+i)^{\frac{1}{m}}+(1+i)^{\frac{2}{m}}+\ldots+(1+i)^{n-\frac{1}{m}}\right\}$$

$$=\frac{1}{m}\cdot\frac{(1+i)^n-1}{(1+i)^{\frac{1}{m}}-1},$$

$$\text{Present value of do.} = \frac{1}{m}\left\{v^{\frac{1}{m}} + v^{\frac{2}{m}} + \ldots + v^n\right\}$$

$$= \frac{1}{m} \cdot \frac{1-v^n}{(1+i)^{\frac{1}{m}}-1}.$$

These formulas are the same as those obtained in Article 33, as they should be.

(See Example 5 of the Illustrations of this Chapter, in Chapter V.)

If we suppose $m=\infty$—that is, the annuity payable momently,—then the denominator of the above fractions, $m\{(1+i)^{\frac{1}{m}}-1\}$, takes the form $\infty \times 0$. We have, however,

$$m\{(1+i)^{\frac{1}{m}}-1\} = \frac{(1+i)^{\frac{1}{m}}-1}{\frac{1}{m}},$$

and the limiting value of this fraction can be shown to be $\log_{\epsilon}(1+i)$. (For demonstration, see De Morgan's "Differential and Integral Calculus," p. 56, and Note on p. 64 of this Chapter.)

Thus the amount of an annuity of 1 payable by momently instalments, the interest at rate i being convertible yearly $\Big\} = \dfrac{(1+i)^n-1}{\log_{\epsilon}(1+i)}$,

and the present value of do. $= \dfrac{1-(1+i)^{-n}}{\log_{\epsilon}(1+i)}$.

If we put, as before, $\log_{\epsilon}(1+i)=\delta$, these formulas become respectively

$$\frac{\epsilon^{n\delta}-1}{\delta} \text{ and } \frac{1-\epsilon^{-n\delta}}{\delta},$$

being the same as those already obtained in Article 36.

(41) *Annuity payable* k *times a year, and Interest convertible* m *times a year :—*

Let n be the number of years the annuity has to run, and let the annual payment be 1, payable by instalments of $\frac{1}{k}$ at the end of each interval of $\frac{1}{k}$th of a year; and let the nominal rate of

interest be x convertible at the end of each interval of $\frac{1}{m}$th of a year.

Then $\left(1+\frac{x}{m}\right)^m$ = amount of 1 in a year $=1+i$, say.

Now as 1 amounts to $1+i$ at the end of a year, it will amount to $(1+i)^{\frac{1}{k}}$ at the end of $\frac{1}{k}$th of a year (Art. 4), to $(1+i)^{\frac{2}{k}}$ at the end of $\frac{2}{k}$ths of a year, and so on. Hence we have, each payment of annuity being $\frac{1}{k}$,

Amount of annuity for n years

$$=\frac{1}{k}\left\{1+(1+i)^{\frac{1}{k}}+(1+i)^{\frac{2}{k}}+\ldots+(1+i)^{n-\frac{1}{k}}\right\}$$

$$=\frac{1}{k}\cdot\frac{(1+i)^n-1}{(1+i)^{\frac{1}{k}}-1},$$

and the present value of the annuity

$$=\frac{1}{k}\left\{v^{\frac{1}{k}}+v^{\frac{2}{k}}+\ldots+v^n\right\}=\frac{1}{k}\,\frac{1-v^n}{(1+i)^{\frac{1}{k}}-1}.$$

Substituting for $1+i$ its value, $\left(1+\frac{x}{m}\right)^m$, these formulas become respectively

$$\text{Amount of an annuity for } n \text{ years} = \frac{1}{k}\cdot\frac{\left(1+\frac{x}{m}\right)^{mn}-1}{\left(1+\frac{x}{m}\right)^{\frac{m}{k}}-1},$$

$$\text{and present value of do.} = \frac{1}{k}\cdot\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{\left(1+\frac{x}{m}\right)^{\frac{m}{k}}-1}.$$

We might have proceeded thus. Let y = nominal rate of interest convertible k times a year (that is, at same periods as annuity is payable), which gives the same effective yearly rate i as the nominal rate x convertible m times a year. Then we have

$$\left(1+\frac{x}{m}\right)^m=\left(1+\frac{y}{k}\right)^k=1+i;$$

$$\therefore \quad y=\{(1+i)^{\frac{1}{k}}-1\}k,$$

and the formulas obtained in Art. 34 would be at once applicable. If we assume $m=k$, the formulas above obtained, become the same as those of Art. (34).

If the interest is convertible momently, and the annuity payable k times a year, then $m=\infty$, and $\left(1+\frac{x}{m}\right)^m=1+i=\epsilon^\delta$. (Art. 8.)

$\therefore$ Amount of an annuity for n years payable k times a year, interest at the rate δ convertible momently $= \frac{1}{k}\cdot\frac{\epsilon^{n\delta}-1}{\epsilon^{\frac{\delta}{k}}-1}$,

and present value of do. $= \frac{1}{k}\cdot\frac{1-\epsilon^{-n\delta}}{\epsilon^{\frac{\delta}{k}}-1}$.

If the interest is convertible m times a year, and the annuity payable momently, then

$$k=\infty, \text{ and } k\left(1+\frac{x}{m}\right)^{\frac{m}{k}}-1=k(1+i)^{\frac{1}{k}}-1$$

$$=\frac{(1+i)^{\frac{1}{k}}-1}{\frac{1}{k}}$$

$$=\log_\epsilon(1+i) \quad \text{(Art. 40.)}$$

$$=\delta.$$

$\therefore$ Amount of an annuity for n years payable momently, interest at the rate x convertible m times a year, the effective rate of interest being i $= \frac{\left(1+\frac{x}{m}\right)^{mn}-1}{\delta}$

$$=\frac{(1+i)^n-1}{\delta}=\frac{\epsilon^{n\delta}-1}{\delta},$$

and present value of do. $= \frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{\delta}$

$$=\frac{1-(1+i)^{-n}}{\delta}=\frac{1-\epsilon^{-n\delta}}{\delta}.$$

These formulas are the same as those of Art. (36).

(42) Let it be required to find the amount and present value of an annuity when interest at the nominal rate 5 per cent. is convertible, and the instalments of the annuity payable quarterly, and the annuity to run for 25 years.

Numerical illustrations of the preceding Articles.

Here $\left.\begin{aligned} x &= \cdot 05 \\ m &= 4 \\ n &= 25 \end{aligned}\right\} \quad \therefore \quad \frac{x}{m} = \cdot 0125$ and $mn = 100$.

Art. (33).

We have amount of annuity $= \dfrac{(1 \cdot 0125)^{100} - 1}{\cdot 05}$.

Now, $\log 1 \cdot 0125 = \cdot 005395 \therefore (1 \cdot 0125)^{100} = \log^{-1} \cdot 5395$

$= 3 \cdot 4634$ (approximately)

$\therefore$ Amount of annuity $= \dfrac{2 \cdot 4634}{\cdot 05} = 49 \cdot 268$

$$\begin{aligned} \log 49 \cdot 268 &= 1 \cdot 692565 \\ \log 3 \cdot 4634 &= 0 \cdot 539503 \\ \hline \log 14 \cdot 225 &= \cdot 153062 \end{aligned}$$

Again, present value of annuity $= \dfrac{49 \cdot 268}{3 \cdot 4634}$

$= 14 \cdot 225$

Let us make use of the Table I. of Chapter I. to check the values just obtained. On reference to that table it will be found that if the nominal rate of interest be 5 per-cent payable quarterly, the equivalent effective yearly rate is 5·094534 per-cent.

Thus, we have Art. (33) formulas (1), (2), (3), and (4),

$$\text{Amount of annuity} = \frac{(1 \cdot 05094534)^{25} - 1}{\cdot 05}$$

$$\text{Present value} \;\; „ \;\; = \frac{1 - (1 \cdot 05094534)^{25}}{\cdot 05}$$

Now, $\log 1 \cdot 05094534 = \cdot 021580$ (to 6 decimal places)

$\therefore \; 25 \times \;\; „ \;\; = \cdot 5395 = \log (1 \cdot 05094534)^{25}$

Hence, $(1 \cdot 05094534)^{25} = 3 \cdot 4634$ (approximately)

$= (1 \cdot 0125)^{100}$ as found above.

Again, $\log 1{\cdot}05094534 \quad = {\cdot}021580$

$$\therefore \quad \log (1{\cdot}05094534)^{\frac{1}{4}} = {\cdot}005395$$

$$\therefore \quad (1{\cdot}05094534)^{\frac{1}{4}} = 1{\cdot}0125$$

$$\therefore \quad 4\{(1{\cdot}05094534)^{\frac{1}{4}} - 1\} = {\cdot}05$$

which is an example of the formula $m\{(1+i)^{\frac{1}{m}} - 1\} = x$.

Numerical illustration of Art. (34).

As a numerical illustration of Art. (34), let us take the following:—An annuity payable quarterly, and interest convertible quarterly, the annuity to run for 5 years, with interest at the nominal rate of 5 per-cent.

$$\text{By Table I. of Chapter I. } \left(1+\frac{{\cdot}05}{4}\right)^4 = 1{\cdot}05094534.$$

Let us first calculate the values of the annuity from the formula.

$$\text{Value of annuity} = \frac{1-(1{\cdot}05094534)^{-5}}{{\cdot}05}.$$

We have $\log 1{\cdot}05094534 \quad = {\cdot}021580$ (to 6 decimal places)

$$\therefore \quad \log (1{\cdot}05094534)^{-5} = - \quad {\cdot}107900$$

$$= \quad \bar{1}{\cdot}892100$$

$$= \log {\cdot}78001$$

$$\therefore \quad \text{Value of annuity} = \frac{1-{\cdot}78001}{{\cdot}05} = \frac{{\cdot}21999}{{\cdot}05}$$

$$= 4{\cdot}3998.$$

We have also $\log {\cdot}05094534 = \bar{2}{\cdot}707105$

$$\log {\cdot}05 \qquad = \bar{2}{\cdot}698970$$

$$\therefore \quad \log \frac{{\cdot}05094534}{{\cdot}05} = {\cdot}008135.$$

Let us now proceed to ascertain the amount of capital redeemed in each quarterly payment.

The following table shows the process throughout:—

p	$\log v^{5-\frac{p}{4}}$	$v^{5-\frac{p}{4}}$	$\frac{1}{4}v^{5-\frac{p}{4}}$ = Amount redeemed in $(p+1)$th quarter.	Total redeemed in $(p+1)$th year.	$\log v^{5-\frac{p}{4}} + \log \frac{\cdot 05094534}{\cdot 05}$	$v^{5-\frac{p}{4}} \times \frac{\cdot 05094534}{\cdot 05}$
0	$\bar{1}$·892100	·78001	·19500	...	$\bar{1}$·900235	·79476
1	·897495	·78976	·19744	...	...	...
2	·902890	·79963	·19991	...	...	...
3	·908285	·80963	·20241	·79476	...	...
4	·913680	·81975	·20494	...	·921815	·83525
5	·919075	·82999	·20749	...	...	...
6	·924470	·84037	·21009	...	...	...
7	·929865	·85087	·21272	·83524	...	...
8	·935260	·86151	·21538	...	·943395	·87780
9	·940655	·87227	·21807	...	...	...
10	·946050	·88319	·22079	...	...	...
11	·951445	·89422	·22355	·87779	...	...
12	·956840	·90540	·22635	...	·964975	·92252
13	·962235	·91672	·22918	...	...	...
14	·967630	·92818	·23204	...	...	...
15	·973025	·93978	·23494	·92251	...	...
16	·978420	·95152	·23788	...	·986555	·96951
17	·983815	·96342	·24085	...	...	...
18	·989210	·97548	·24387	...	...	...
19	·994605	·98766	·24691	·96951	...	...
				4·39981		

From these figures it will be seen how much is redeemed at the end of each quarter of the 5 years: thus

At the end of the first quarter the amount is ·19500

" " second " " ·19744,

and so on.

The entire amount redeemed in the four quarters of the first year is seen to be the sum of the first four values of $\frac{1}{4}v^{5-\frac{p}{4}}$, *i.e.*, ·79476.

The entire amount redeemed in the four quarters of the second year is seen to be the sum of the second four values of $\frac{1}{4}v^{5-\frac{p}{4}}$, *i.e.*, ·83524; and so on.

It is further to be observed that the sum of the first four values of $\frac{1}{4}v^{5-\frac{p}{4}}$, is equal to $v^5 \times \dfrac{\cdot 05094534}{\cdot 05}$,

and that the sum of the second four values is equal to $v^4 \times \dfrac{\cdot 05094534}{\cdot 05}$

„	third	„	„	$v^3 \times \dfrac{\cdot 05094534}{\cdot 05}$
„	fourth	„	„	$v^2 \times \dfrac{\cdot 05094534}{\cdot 05}$
„	fifth	„	„	$v \times \dfrac{\cdot 05094534}{\cdot 05}$

Also, that the total amount redeemed in the course of the five years is equal to the original value of the annuity, as it should be, *i.e.*, 4·3998.

It will be noted that there are small discrepancies in the last place of decimals, which arise from not employing more decimal places in the logarithmic values.

Numerical illustration of Art. (36).

As a numerical illustration of Art. (36), let us take an annuity for 20 years, the nominal rate of interest being 7 per-cent, and the annuity payable and interest convertible momently: that is, a continuous annuity for 20 years, at a nominal rate of interest of 7 per-cent.

$$n=20,\ \delta=\cdot 07,\ \text{and (Table I., Chapter I.)}\ i=\cdot 07250818.$$

Then we have $\quad$ amount of annuity $=\dfrac{\epsilon^{1\cdot 4}-1}{\cdot 07}$.

Now,
$$\log \epsilon^{1\cdot 4}=1\cdot 4 \log \epsilon = 1\cdot 4 \times \cdot 43429448$$
$$=\cdot 60801227$$
$$=\log 4\cdot 0552$$
$$\therefore \quad \frac{\epsilon^{1\cdot 4}-1}{\cdot 07}=\frac{3\cdot 0552}{\cdot 07}$$
$$=43\cdot 6457$$

Otherwise, since the effective rate of interest is known, we might proceed thus:

$$\text{Amount of annuity}=\frac{(1\cdot 07250818)^{20}-1}{\cdot 07250818} \times \frac{\cdot 07250818}{\cdot 07};$$

but as existing annuity tables would not be found to give the value of the first factor, it would have to be calculated.

Again, $$\text{present value of annuity} = \frac{1-\epsilon^{-1\cdot4}}{\cdot07}$$

$$= \frac{\epsilon^{1\cdot4}-1}{\cdot07} \times \frac{1}{\epsilon^{1\cdot4}}.$$

We have $$\log\frac{\epsilon^{1\cdot4}-1}{\cdot07} = \log 43\cdot6457$$

$$= 1\cdot639942$$

$$\log\epsilon^{1\cdot4} = \frac{\cdot608012}{1\cdot031930}$$

$$\therefore \quad \text{present value of annuity} = \log^{-1} 1\cdot031930$$

$$= 10\cdot7629.$$

Numerical illustration of Art. (41). As a numerical illustration of Article (41), let us take the case of an annuity for 20 years, where interest at the nominal rate of 5 per-cent is convertible quarterly, and instalments of annuity are payable at the end of every 12th part of a year—that is, monthly.

Then $x = \cdot05$, $m = 4$, and $k = 12$.

We have $\left(1+\frac{\cdot05}{4}\right)^4 = (1\cdot0125)^4 = 1\cdot05094534$ (Table I., Chap. I.)

$$\text{Log } 1\cdot05094534 = \cdot021580 \quad \text{(to 6 places of decimals)}$$

$$\therefore \log(1\cdot05094534)^{20} = \cdot43160 = \log 2\cdot7014$$

$$\log(1\cdot05094534)^{\frac{1}{12}} = \cdot001798 = \log 1\cdot0041$$

Thus, $$\text{amount of annuity} = \frac{1}{12}\,\frac{2\cdot7014-1}{1\cdot0041-1} = \frac{1\cdot7014}{12\cdot0041}$$

$$= \frac{1\cdot7014}{\cdot0492}$$

$$= 34\cdot58$$

$$\text{and present value will be found to be} = 34\cdot58 \times \frac{1}{2\cdot7014}$$

$$= 12\cdot80.$$

(For further illustrations, see Chapter V.)

NOTE TO ART. (40).

An ordinary algebraical demonstration of the property that $m\{(1+i)^{\frac{1}{m}}-1\}$ has for its limiting value $\log_\epsilon(1+i)$ when $m=\infty$ may be thus shown:—

We have, by the exponential theorem (*see Algebra—Exponential Series*),

$$a^x=1+x\log_\epsilon a+\frac{x^2}{\lfloor 2}(\log_\epsilon a)^2+\frac{x^3}{\lfloor 3}(\log_\epsilon a)^3+\&c.$$

$$=1+x\log_\epsilon a+x\left\{\frac{x}{\lfloor 2}(\log_\epsilon a)^2+\frac{x^2}{\lfloor 3}(\log_\epsilon a)^3+\&c.\right\}$$

$$\therefore\ \frac{a^x-1}{x}=\log_\epsilon a+x\left\{\frac{1}{\lfloor 2}(\log_\epsilon a)^2+\frac{x}{\lfloor 3}(\log_\epsilon a)^2+\&c.\right\}\quad .\ .\quad (1)$$

Now it can be shown by the ordinary rules for the convergence of a series (*see Algebra—Convergence of Series*), that the series inside the brackets on the right-hand side of (1) is convergent whatever finite values a may have, and therefore the product of this series and x vanishes with x. Hence we have

$$\text{Limiting value (when } x=0\text{) of } \frac{a^x-1}{x}=\log_\epsilon a.$$

Let us now put $a=1+i$, and $x=\frac{1}{m}$, so that a is finite and $x=0$ when $m=\infty$, and we have

$$\text{Limiting value (when } m=\infty\text{) of } \frac{(1+i)^{\frac{1}{m}}-1}{\frac{1}{m}} \text{ is } \log_\epsilon(1+i).$$

CHAPTER III.

VARYING ANNUITIES.

ANNUITIES WHERE THE PERIODIC PAYMENTS VARY IN AMOUNT, AND THE PERIODS THEMSELVES ARE CONSTANT OR VARIABLE.

Instances of Varying Annuities.

(43) In the previous chapter we have throughout assumed that the periodic payments of annuity are all of the same amount, that is, 1. It will be seen, however, that we may have cases where the periodic payments vary in amount. Thus, for instance, we may have an annuity where the first payment is 1; the second, 2; the third, 3; and so on, in order. Again, we may have an annuity where the first payment is 100; the second, 95; the third, 90; the fourth, 85; and so on. The first of these would be an annuity where the payments increase in arithmetic progression, and the second an annuity where the payments decrease in arithmetic progression. Similarly, we may have annuities increasing or decreasing in geometric progression, or generally where the payments follow some law of progression; and we may have annuities where the payments follow no law of progression at all.

Let u_1, u_2, u_3, &c., denote the amounts payable respectively at the end of 1 year, 2 years, 3 years, and so on. Then, if there be n such payments, and i be the rate of interest, we have,

$$\text{Present value of payments} = u_1 v + u_2 v^2 + u_3 v^3 + \ .\ .\ .\ + u_n v^n.$$

On the methods to be employed for the determination of the present value of Varying Annuities.

Where the quantities u_1, u_2, &c., are regulated according to some known law, a formula for the summation of the above series may frequently be obtained. Indeed, under such circumstances, the problem would become an ordinary application of the methods for the Summation of Series.

Simple algebraical illustrations.

(44) In many cases this series can be summed by ordinary algebraical methods. Thus, for instance, if the law of progression of u_1, u_2, u_3, &c., is arithmetic, common difference, say, d, then we have, denoting the sum to n terms by S,

$$S=u_1v+(u_1+d)v^2+(u_1+2d)v^3+\ldots+\{u_1+\overline{n-2}d\}v^{n-1}+\{u_1+\overline{n-1}d\}v^n;$$

$$\therefore\quad vS=u_1v^2+(u_1+d)v^3+\ldots+\{u_1+\overline{n-3}d\}v^{n-1}+\{u_1+\overline{n-2}d\}v^n$$
$$+\{u_1+\overline{n-1}d\}v^{n+1};$$

$$\therefore\quad (1-v)S=u_1v+d(v^2+v^3+\ldots+v^n)-\{u_1+\overline{n-1}d\}v^{n+1};$$

$$\therefore\quad S=\frac{u_1v-\{u_1+(n-1)d\}v^{n+1}}{1-v}+\frac{dv^2(1-v^{n-1})}{(1-v)^2}.$$

As another example, let it be required to find the present value of an annuity where the successive payments are $1^2, 2^2, 3^2 \ldots n^2$. If S denote the present value of this annuity, i being the rate of interest, then

$$S=1^2v+2^2v^2+3^2v^3+\ldots+n^2v^n;$$

$$\therefore\quad Sv=1^2v^2+2^2v^3+\ldots+(n-1)^2v^n+n^2v^{n+1};$$

$$\therefore\quad S(1-v)=v+(2^2-1^2)v^2+(3^2-2^2)v^3+\ldots+\{n^2-(n-1)^2\}v^n-n^2v^{n+1}$$
$$=v+3v^2+5v^3+\ldots+(2n-1)v^n-n^2v^{n+1};$$

$$\therefore\quad Sv(1-v)=v^2+3v^3+\ldots+\{2(n-1)-1\}v^n+(2n-1)v^{n+1}-n^2v^{n+2};$$

$$\therefore\quad S(1-v)^2=v+2(v^2+v^3+\ldots+v^n)-(n^2+2n-1)v^{n+1}+n^2v^{n+2}$$
$$=v+\frac{2v^2(1-v^{n-1})}{1-v}-(n^2+2n-1)v^{n+1}+n^2v^{n+2}$$
$$=\frac{v-v^2+2v^2-2v^{n+1}-v^{n+1}\{n^2(1-v)^2+2n(1-v)\}+(1-v)v^{n+1}}{1-v}$$
$$=\frac{v(1+v)-v^{n+1}\{n^2(1-v)^2+2n(1-v)+1\}-v^{n+2}}{1-v};$$

$$\therefore\quad S=\frac{v(1+v)-v^{n+1}\{n(1-v)+1\}^2-v^{n+2}}{(1-v)^3}.$$

The annuity-values just obtained are not in a shape readily available for arithmetical calculation; and it will be frequently found to be the simplest plan not to attempt to obtain a formula which shall represent the value of the sum of the series,—that is, of the annuity,—but to calculate independently and directly the value of each payment of the annuity, and sum the results.

On a general formula by Makeham for the determination of the present values and amounts of Varying Annuities.

(45) In cases where the law (if any) connecting the terms of the series is not ascertained, or where no such law exists, summation may be frequently effected in the following manner:—

Let Δu_1, $\Delta^2 u_1$, $\Delta^3 u_1 \ldots$ denote the successive differences of u_1, then we have (see any treatise on Finite Differences)

$$u_1 = u_1$$
$$u_2 = u_1 + \Delta u_1$$
$$u_3 = u_1 + 2\Delta u_1 + \Delta^2 u_1$$
$$\cdots\cdots$$
$$u_n = u_1 + (n-1)\Delta u_1 + \frac{(n-1)(n-2)}{1.2}\Delta^2 u_1 + \ldots + \Delta^{n-1} u_1.$$

Hence the series $u_1 v + u_2 v^2 + u_3 v^3 + \ldots + u_n v^n$ will be equivalent to the following:—

$$\begin{aligned} & u_1(v + v^2 + v^3 + \ldots + v^n) \\ + \; & \Delta u_1\{v^2 + 2v^3 + \ldots + (n-1)v^n\} \\ + \; & \Delta^2 u_1\left\{v^3 + 3v^4 + \ldots\ldots + \frac{(n-1)(n-2)}{1.2} v^n\right\} \\ + \; & \ldots\ldots\ldots \\ + \; & \Delta^r u_1\left\{v^{r+1} + (r+1)v^{r+2} + \ldots + \frac{(n-1)(n-2)\ldots(n-r)}{1.2.3\ldots r} v^n\right\} \\ + \; & \ldots\ldots \\ + \; & \Delta^{n-1} u_1 v^n. \end{aligned}$$

Now, taking the general term

$$\Delta^r u_1\left\{v^{r+1} + (r+1)v^{r+2} + \ldots + \frac{(n-1)(n-2)\ldots(n-r)}{1.2.3\ldots r} v^n\right\},$$

we see that the coefficient of v^x will be

$$\frac{(x-1)(x-2)\ldots(x-r)}{1.2.3\ldots r};$$

and if this be denoted by $\overset{r+1}{[x]}$, the general term

$$\Delta^r u_1\left\{v^{r+1}+(r+1)v^{r+2}+\ldots+\frac{(n-1)(n-2)\ldots(n-r)}{1.2.3\ldots r}v^n\right\}$$

may be written $\Delta^r u_1\{\overset{r+1}{[1]}v+\overset{r+1}{[2]}v^2+\ldots+\overset{r+1}{[n]}v^n\}$.

The quantity inside the brackets represents the present value of an annuity for n years, of which the xth payment, denoted by $\overset{r+1}{[x]}$, is $\dfrac{(x-1)(x-2)\ldots(x-r)}{1.2.3\ldots r}$. If we denote this present value by $\overset{r+1}{V}_n$, the above expression for the value of an annuity whose successive payments are u_1, u_2, $u_3\ldots u_n$, will become

$$u_1\overset{1}{V}_n+\Delta u_1\overset{2}{V}_n+\Delta^2 u_1\overset{3}{V}_n+\ldots\quad\ldots\ldots\quad(A)$$

if we denote by $\overset{1}{V}_n$ the value of an annuity for n years, where the successive payments are unity.

We have now to obtain a formula for the calculation of $\overset{1}{V}_n$, $\overset{2}{V}_n$, $\overset{3}{V}_n$, &c.

We have, by definition,

$$\overset{r}{[x]}=\frac{(x-1)(x-2)\ldots(x-r+1)}{1.2.3\ldots(r-1)}$$

$$\overset{r+1}{[x]}=\frac{(x-1)(x-2)\ldots(x-r)}{1.2.3\ldots r}=\frac{x-1}{r}[\overset{r}{x-1}].$$

Also

$$[\overset{r+1}{x-1}]=\frac{(x-2)(x-3)\ldots\ldots(x-r-1)}{1.2.3\ldots r}=\frac{x-r-1}{r}[\overset{r}{x-1}]$$

$$=\frac{x-1}{r}[\overset{r}{x-1}]-[\overset{r}{x-1}];$$

$$\therefore\quad[\overset{r+1}{x-1}]+[\overset{r}{x-1}]=\frac{x-1}{r}[\overset{r}{x-1}]$$

$$=\overset{r+1}{[x]}.$$

Hence we see that the xth term of what we will call the $(r+1)$th series, is equal to the sum of the $(x-1)$th terms of that series, and of the rth series.

Again, since

$$[\overset{r}{x-1}]=[\overset{r+1}{x}]-[\overset{r+1}{x-1}]$$

$$[\overset{r}{x-2}]=[\overset{r+1}{x-1}]-[\overset{r+1}{x-2}],$$

$$\&c. = \&c.$$

$\therefore$ by addition,

$$[\overset{r}{x-1}]+[\overset{r}{x-2}]+[\overset{r}{x-3}]+\ldots=[\overset{r+1}{x}],$$

since the first term of the $(r+1)$th series is zero.

Thus, the xth term of the $(r+1)$th series is equal to the sum of the first $(x-1)$ terms of the rth series.

Let us denote unity by $\overset{0}{V}_n$. We shall now proceed to show that the connection between $\overset{r-1}{V}_n$ and $\overset{r}{V}_n$, which denote the present values of annuities for n years where the payments made at the end of any year, say the pth year, are respectively,

$$\frac{(p-1)(p-2)\ldots.(p-r+2)}{1.2\ldots(r-2)} \text{ and } \frac{(p-1)(p-2)\ldots.(p-r+1)}{1.2\ldots.(r-1)},$$

is

$$\overset{r}{V}_n=\frac{\overset{r-1}{V}_n-[\overset{r}{n+1}]v^n}{i}.$$

We have

$$\overset{r}{V}_n=[\overset{r}{1}]v+[\overset{r}{2}]v^2+\ldots+[\overset{r}{n}]v^n,$$

$$\overset{r-1}{V}_n=[\overset{r-1}{1}]v+[\overset{r-1}{2}]v^2+\ldots+[\overset{r-1}{n}]v^n;$$

$$\therefore \overset{r}{V}_n+\overset{r-1}{V}_n=[\overset{r}{2}]v+[\overset{r}{3}]v^2+\ldots+[\overset{r}{n+1}]v^n$$

$$=\{[\overset{r}{1}]v+[\overset{r}{2}]v^2+[\overset{r}{3}]v^3+\ldots+[\overset{r}{n+1}]v^{n+1}\}(1+i),$$

$$=\overset{r}{V}_{n+1}(1+i);$$

since $[\overset{r}{1}]$ is zero for all values of r greater than 1.

But

$$\overset{r}{V}_{n+1}=\overset{r}{V}_n+[\overset{r}{n+1}]v^{n+1};$$

$$\therefore \overset{r}{V}_n+\overset{r-1}{V}_n=\overset{r}{V}_n(1+i)+[\overset{r}{n+1}]v^n;$$

$$\therefore i\overset{r}{V}_n=\overset{r-1}{V}_n-[\overset{r}{n+1}]v^n;$$

$$\therefore \overset{r}{V}_n=\frac{\overset{r-1}{V}_n-[\overset{r}{n+1}]v^n}{i} \quad \ldots\ldots \quad \text{(B)}$$

Similarly, if we denote the amount of an annuity for n years whose pth payment is $[\overset{r}{p}]$ by $\overset{r}{A}_n$, and denote $(1+i)^n$ by $\overset{0}{A}_n$, we shall have

$$\overset{r}{A}_n = \frac{\overset{r-1}{A}_n - [\overset{r}{n+1}]}{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

Thus, the values and amounts of annuities corresponding to any value of r can be obtained, when those corresponding to $r-1$ are known.

The following table of values of $[\overset{r}{x}]$ gives the xth payment of the annuity corresponding to a given value of r:—

x	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0
4	1	3	3	1	0	0	0	0
5	1	4	6	4	1	0	0	0
6	1	5	10	10	5	1	0	0
7	1	6	15	20	15	6	1	0
8	1	7	21	35	35	21	7	1

Taking the formula (B), we have, putting r successively equal to 1, 2, 3, 4, &c.,

$$\overset{1}{V}_n = \frac{1-v^n}{i},$$

$$\overset{2}{V}_n = \frac{\overset{1}{V}_n - nv^n}{i},$$

$$\overset{3}{V}_n = \frac{\overset{2}{V}_n - \frac{n.n-1}{2} v^n}{i},$$

$$\overset{4}{V}_n = \frac{\overset{3}{V}_n - \frac{n.n-1.n-2}{2.3} v^n}{i},$$

&c. = &c.

Similarly
$$\overset{1}{A}_n=\frac{(1+i)^n-1}{i},$$

$$\overset{2}{A}_n=\frac{\overset{1}{A}_n-n}{i},$$

$$\overset{3}{A}_n=\frac{\overset{2}{A}_n-\frac{n.n-1}{2}}{i},$$

$$\overset{4}{A}_n=\frac{\overset{3}{A}_n-\frac{n.n-1.n-2}{2.3}}{i},$$

&c.= &c.

Thus, reverting back to the formula (A), we see that, since

$$u_1v+u_2v^2+\ldots+u_nv^n=u_1\overset{1}{V}_n+\Delta u_1\overset{2}{V}_n+\Delta^2u_1\overset{3}{V}_n+\ldots$$

when the values of $\overset{1}{V}_n$, $\overset{2}{V}_n$, &c., have been found in the manner above indicated, the present value of the series of annual payments $u_1, u_2, u_3 \ldots u_n$, may be directly found.

And the same method would apply to finding the sum to which the same annual payments would accumulate at the end of the term, or this may be deduced from the above by simple multiplication by $(1+i)^n$.

Numerical illustration of the general formula of Art. (45).

(46) It will be sufficient to give one numerical illustration of the method just explained.

Let the annuity be one to run for 40 years, the successive payments being 417500, 424680, 431620, 438320, &c., and the rate of interest 5 per-cent.

We have

	Δ	Δ^2	Δ^3
$u_1=417500$			
	$+7180$		
$u_2=424680$		-240	
	$+6940$		0
$u_3=431620$		-240	
	$+6700$		
$u_4=438320$			

Here it will be seen that

$$\Delta u_1=+7180$$
$$\Delta^2u_1=-240$$
$$\Delta^3u_1=0$$

The third and higher differences vanish.

Hence we have

$$\text{Present value of annuity (A)} = u_1\overset{1}{V}_{40} + \Delta u_1\overset{2}{V}_{40} + \Delta^2 u_1\overset{3}{V}_{40}$$

$$= 417500\overset{1}{V}_{40} + 7180\overset{2}{V}_{40} - 240\overset{3}{V}_{40} \quad . \quad (1)$$

We have now to calculate $\overset{1}{V}_{40}$, $\overset{2}{V}_{40}$, $\overset{3}{V}_{40}$, by means of (B).

$\log(1{\cdot}05)^{-40} = \cdot 152428037200$

$(1{\cdot}05)^{-40} = \cdot 14204568$

$40(1{\cdot}05)^{-40} = 5{\cdot}6818272$

$40(1{\cdot}05)^{-40} \times \frac{39}{2} = 113{\cdot}636544$

$\quad - \ 2{\cdot}840914$

$= 110{\cdot}795630$

$1-(1{\cdot}05)^{-40} = \cdot 85795432$

$\therefore \overset{1}{V}_{40} = 17{\cdot}1590864$

$\therefore \overset{1}{V}_{40} - 40(1{\cdot}05)^{-40} = 17{\cdot}1590864$

$\quad - \ 5{\cdot}6818272$

$= 11{\cdot}4772592$

$\therefore \overset{2}{V}_{40} = 229{\cdot}545184$

$\quad 110{\cdot}795630$

$\therefore \overset{2}{V}_{40} - 40(1{\cdot}05)^{-40} \times \frac{39}{2} = 118{\cdot}749554$

$\therefore \overset{3}{V}_{40} = 2374{\cdot}99108$

Having obtained the values of $\overset{1}{V}_{40}$, $\overset{2}{V}_{40}$, $\overset{3}{V}_{40}$, we have now to insert these in (1):

log 417500 $= 5{\cdot}6206565$	log 7180 $= 3{\cdot}8561244$	log 240 $= 2{\cdot}3802112$
„ 17·159086 $= 1{\cdot}2344943$	„ 229·545184 $= 2{\cdot}3608683$	„ 2374·99108 $= 3{\cdot}3756620$
log 7,163,922 $= 6{\cdot}8551508$	log 1,648,135 $= 6{\cdot}2169927$	log 569,998 $= 5{\cdot}7558732$

Hence

present value of given annuity $= 7{,}163{,}922 + 1{,}648{,}135 - 569{,}998$

$= 8{,}812{,}057 - 569{,}998$

$= 8{,}242{,}059.$

The method just explained and illustrated is due to Mr. W. M. Makeham, and is described by him at length in a paper "On the Theory of Annuities Certain", published in vol. xiv. of the *Journal of the Institute of Actuaries.*

As to the applicability of the formula of Art. (45).

Where the successive payments of the annuity are of such a nature that only a few orders of differences are required, Mr. Makeham's method will be found of great practical utility.

(47) Another formula which has been employed for the summation of the series, On another general formula

$$u_1v+u_2v^2+u_3v^3+\ldots+u_nv^n,$$

is as follows:

$$u_1v+u_2v^2+u_3v^3+\ldots+u_nv^n=\frac{u_1}{i}+\frac{\Delta u_1}{i^2}+\frac{\Delta^2 u_1}{i^3}+\ldots$$

$$-v^n\left(\frac{u_{n+1}}{i}+\frac{\Delta u_{n+1}}{i^2}+\frac{\Delta^2 u_{n+1}}{i^3}+\ldots\right)\quad . \quad . \quad . \quad . \quad (D)$$

The theorem from which this formula is deduced is as follows:— If u_n be a rational and integral function of n, so that the differences of u_1 ultimately vanish, then

$$u_1x+u_2x^2+u_3x^3+\ldots ad\ inf.=\frac{x}{1-x}u_1+\frac{x^2}{(1-x)^2}\Delta u_1$$

$$+\frac{x^3}{(1-x)^3}\Delta^2 u_1+\ldots \quad . \quad . \quad . \quad (1)$$

Similarly, $\quad u_{n+1}x^{n+1}+u_{n+2}x^{n+2}+u_{n+3}x^{n+3}+\ldots ad\ inf.$

$$=x^n(u_{n+1}x+u_{n+2}x^2+\ldots ad\ inf.)$$

$$=x^n\left(\frac{x}{1-x}u_{n+1}+\frac{x^2}{(1-x)^2}\Delta u_{n+1}+\frac{x^3}{(1-x)^3}\Delta^2 u_{n+1}+\&c.\right)\quad . \quad . \quad (2)$$

$\therefore$ Subtracting (2) from (1) we have,

$$u_1x+u_2x^2+u_3x^3+\ldots+u_nx^n=\frac{x}{1-x}u_1+\frac{x^2}{(1-x)^2}\Delta u_1$$

$$+\frac{x^3}{(1-x)^3}\Delta^2 u_1+\ldots-x^n\left(\frac{x}{1-x}u_{n+1}+\frac{x^2}{(1-x)^2}\Delta u_{n+1}+\&c.\right).$$

Now, in our case, $\quad x=v=\frac{1}{1+i}\qquad\therefore\quad\frac{x}{1-x}=\frac{1}{i};$

$$\therefore\quad u_1v+u_2v^2+u_3v^3+\ldots+u_nv^n=\frac{u_1}{i}+\frac{\Delta u_1}{i^2}+\frac{\Delta^2 u_1}{i^3}+\&c.$$

$$-v^n\left(\frac{u_{n+1}}{i}+\frac{\Delta u_{n+1}}{i^2}+\frac{\Delta^2 u_{n+1}}{i^3}+\&c.\right),$$

which is the formula (D).*

The theorem here applied will be found demonstrated on page 224 of De Morgan's "Differential and Integral Calculus" (published in 1842), and the general theorem of which it is a particular case, on pages 239 and 240 of the same work.

* This formula may be deduced directly from Mr. Makeham's formula (A). See Chapter VI., Art. (80).

Mr. Peter Gray has, in vol. xiv. of the *Journal of the Institute of Actuaries*, called attention to this formula, and its application to the determination of the values of annuities of a nature similar to that under consideration.

Numerical illustration of Art. (47).

(48) As an instance of its application, the example above selected may be taken; viz., an annuity to run for 40 years, the rate of interest being 5 per-cent, and the successive payments being 417500, 424680, 431620, 438320, &c.

As before, we have

$$u_1 = 417500,$$
$$\Delta u_1 = +7180,$$
$$\Delta^2 u_1 = -\ 240,$$
$$\Delta^3 u_1 = \quad 0.$$

We have still to calculate u_{41}, Δu_{41}, and $\Delta^2 u_{41}$.

$$\text{Now} \qquad u_{1+m} = u_1 + m\Delta u_1 + \frac{m \,.\, m-1}{2}\Delta^2 u_1 + \ldots$$
$$= u_1 + 7300m - 120m^2.$$

Putting $m=40$, 41, and 42 successively, we get

$$\begin{array}{lcc} & \Delta & \Delta^2 \\ u_{41} = 517{,}500 & & \\ & -2420 & \\ u_{42} = 515{,}080 & & -240 \\ & -2660 & \\ u_{43} = 512{,}420 & & \end{array}$$

From these we get $\Delta b_{41} = -2420$, and $\Delta^2 b_{41} = -240$.

Thus the value of the annuity will be given by the expression

$$\frac{417500}{i} + \frac{7180}{i^2} - \frac{240}{i^3} - v^{40}\left(\frac{517500}{i} - \frac{2420}{i^2} - \frac{240}{i^3}\right) \quad . \quad . \quad (1)$$

Calling these 6 terms A, B, C, D, E, F, respectively, the value of the annuity as given in (1) will be represented by

$$A + B + E + F - (C + D).$$

We have $\log i = \log \cdot 05 = \bar{2}\cdot 6989700 \quad . \quad . \quad (1)$

$\therefore \quad \log i^2 = \bar{3}\cdot 3979400 \quad . \quad . \quad (2)$

$\log i^3 = \bar{4}\cdot 0969100 \quad . \quad . \quad (3)$

Also $\log(1{\cdot}05)^{-40} = \log v^{40} = \bar{1}{\cdot}1524280$

$$\therefore \quad \log\frac{v^{40}}{i} = 0{\cdot}4534580 \quad . \quad . \quad (4)$$

$$\log\frac{v^{40}}{i^2} = 1{\cdot}7544880 \quad . \quad . \quad (5)$$

$$\log\frac{v^{40}}{i^3} = 3{\cdot}0555180 \quad . \quad . \quad (6)$$

A	B	C
log 417500 = 5·6206565	log 7180 = 3·8561244	log 240 = 2·3802112
(1) = $\bar{2}$·6989700	(2) = $\bar{3}$·3979400	(3) = $\bar{4}$·0969100
6·9216865	6·4581844	6·2833012
∴ A = 8350000	B = 2872000	C = 1920000

D	E	F
log 517500 = 5·7139104	log 2420 = 3·3838154	log 240 = 2·3802112
(4) = 0·4534580	(5) = 1·7544880	(6) = 3·0555180
6·1673684	5·1383034	5·4357292
∴ D = 1470173	E = 137500	F = 272728

A = 8350000
B = 2872000
E = 137500
F = 272728
11632228
− 3390173
8242055

C = 1920000
D = 1470173
3390173

Hence we see that the formula (D) gives for the value of the annuity 8,242,055, the value obtained by the use of the formula (A) being 8,242,059,—results practically identical.

Observations on the formulas obtained in previous Articles.

(49) It will be observed that the formulas (A) and (D) both depend, as regards their practical utility, upon the series of annual payments being such that the differences of u_1 vanish after a few orders of differences.

Taking the formula (D), it will be seen that if the number of payments n is infinite, the series simply becomes

$$\frac{u_1}{i} + \frac{\Delta u_1}{i^2} + \frac{\Delta^2 u_2}{i^3} + \ldots \quad . \quad . \quad . \quad . \quad (E)$$

which will then represent the value of a perpetuity where the annual payments are u_1, u_2, u_3, &c.

The formula (A) may be shown to resolve itself into the same expression; and for a demonstration of this reference should be made to Mr. Makeham's paper above alluded to. Mr. Makeham further points out that caution has to be exercised in taking sufficient orders of differences, otherwise, in consequence of the magnitude of the co-efficients, the error caused by the neglected differences may be considerable.

Other cases of Varying Annuities.

(50) Instead of the series of payments of the annuity being of varying amounts made in successive years, we may have an annuity where the payments vary in amount, but are only made at intervals of time—say t years apart; or we may even have an annuity with the payments of varying amount, and the periods at which they are paid also varying. That is, we may have as the series for summation representing the value of the annuity,

$$u_1v^t+u_2v^{2t}+u_3v^{3t}+\ldots,$$

or

$$u_1v^t+u_2v^{t+p}+u_3v^{t+p+q}+\ldots\ldots$$

In the first case, formulas (A) and (D), with suitable modifications, would still hold; but in the latter case it would be generally necessary to obtain the value of the annuity by actual calculation of the value of each payment and summation of the results.

On the separation of the annual payments into payment of interest and repayment of capital respectively.

(51) Let us now proceed to investigate what portion of each payment is respectively payment of interest and repayment of capital in the annuity whose payments are u_1, u_2, u_3, . . . u_n. If we denote the present value of these payments by X, i being the rate of interest, then

$$X=u_1v+u_2v^2+u_3v^3+\ldots+u_nv^n.$$

It will be noted that nothing is assumed as to the values of u_1, u_2, . . . ; but we shall assume, as would almost always happen in practice, that u_1, u_2, . . . are respectively more than sufficient to provide a year's interest on the capital outstanding, so that some portion of X would be repaid each year.

The following table will show how each annual payment is appropriated to payment of interest and repayment of capital respectively:—

Original capital$=X=u_1v+u_2v^2+u_3v^3+\,..\,+u_nv^n$.

Year.	Capital bearing Interest for the Year.	Payment at end of Year.		Payment at end of Year, separated into Interest for Year.		Repayment of Capital.
1	X	u_1	$=$	iX	$+$	u_1-iX
2	$X(1+i)-u_1$	u_2	$=$	$iX(1+i)-iu_1$	$+$	$u_2-i\{X(1+i)-u_1\}$
3	$X(1+i)^2-u_1(1+i)-u_2$	u_3	$=$	$iX(1+i)^2-iu_1(1+i)-iu_2$	$+$	$u_3-i\{X(1+i)^2-u_1(1+i)-u_2\}$
. . . .		. . .		. . . . . .		. . . . . .
n	$X(1+i)^{n-1}-u_1(1+i)^{n-2}-u_2(1+i)^{n-3}-\&c.-u_{n-1}$	u_n	$=$	$i\{X(1+i)^{n-1}-u_1(1+i)^{n-2}-u_2(1+i)^{n-3}-\&c.-u_{n-1}\}$	$+$	$u_n-i\{X(1+i)^{n-1}-u_1(1+i)^{n-2}-\&c.-u_{n-1}\}$

Adding together the repayments of capital made each year, we get the total capital repaid at end of n years to be

$$\left.\begin{array}{l} u_1+u_2+u_3+\,..\,+u_n \\ +iu_1\{1+(1+i)+(1+i)^2+\,..\,+(1+i)^{n-2}\} \\ +iu_2\{1+(1+i)+(1+i)^2+\,..\,+(1+i)^{n-3}\} \\ +\,\ldots \\ +iu_{n-1} \end{array}\right\}-iX\{1+(1+i)+(1+i)^2+\,..\,+(1+i)^{n-1}\}$$

Now

$$iu_1\{1+(1+i)+(1+i)^2+\,..\,+(1+i)^{n-2}\}=iu_1\cdot\frac{(1+i)^{n-1}-1}{i}$$

$$=u_1(1+i)^{n-1}-u_1$$

$$iu_2\{1+(1+i)+(1+i)^2+\,..\,+(1+i)^{n-3}\}=u_2(1+i)^{n-2}-u_2$$

$$\ldots\ldots\ldots \qquad = \qquad \ldots\ldots$$

$$iu_{n-1} \qquad =u_{n-1}(1+i)-u_{n-1}$$

$$iX\{1+(1+i)+(1+i)^2+\,..\,+(1+i)^{n-1}\}=\frac{(1+i)^n-1}{i}iX$$

$$=X(1+i)^n-X.$$

Thus, capital repaid at end of n years

$$=u_1(1+i)^{n-1}+u_2(1+i)^{n-2}+u_3(1+i)^{n-3}+\,..\,+u_{n-1}(1+i)+u_n$$
$$-X(1+i)^n+X$$

$=X$, the original capital.

Hence we see, that just as in the case of ordinary annuities where the annual payments are of the same amount, the capital still unpaid at the end of any year, say the tth year, is

$$=X(1+i)^t-\{u_1(1+i)^{t-1}+u_2(1+i)^{t-2}+\,..\,+u_{t-1}(1+i)-u_t\},$$

that is,

=Original capital accumulated for t years less the sum of the first t payments respectively accumulated for the number of years that have elapsed since they were due.

On the amount of capital repaid at the end of any time when the annual payments remain the same and the rate of interest is different.

(**52**) Taking, as before, X as the capital, and u_1, u_2, u_3, . . . u_n, still as the payments of annuity made at the end of the respective years, it will be seen from what has preceded that the capital still unpaid at the end of any year will depend upon the rate of interest which the outstanding capital is assumed to bear, and will increase as the rate of interest increases.

Thus if i be such a rate of interest that $X=u_1v+u_2v^2+\,..\,+u_nv^n$, then, at the end of n years, we have shown in Art. (51) that the capital X is just then all repaid.

If we assume the capital X to bear interest at k, a rate of

interest greater than i, there will be outstanding at the end of n years capital equal in amount to

$$X(1+k)^n-\{u_1(1+k)^{n-1}+u_2(1+k)^{n-2}+\,..\,+u_{n-1}(1+k)+u_n\},$$

and the present value of this at the same rate of interest k

$$=X-\{u_1v'+u_2v'^2+\,..\,+u_nv'^n\},$$

where $$\frac{1}{(1+k)^n}=v'^n.$$

Thus we see that, if the capital be considered to be the present value of the annual payments $u_1, u_2, \,..\, u_n$ computed at a rate of interest i, then the difference between this present value of the annual payments and the present value of the *same payments* when computed at another rate of interest k, represents the present value of that portion of the capital which would be unpaid at the end of the term when the said capital is to bear interest at the rate k instead of i.

On the separation of the value of the annual payments into value of payments on account of interest and on account of capital respectively, and a general expression for these.

(53) As has been shown in (51), each of the payments $u_1, u_2, u_3, \,..\, u_n$ consists of two parts, one for interest, and the other on account of repayment of capital. In some cases it may prove convenient to determine separately the value of the payments made in respect of each,* the sum of the values of the payments on account of interest and repayment of capital being, of course, equal to the value of the series of payments $u_1, u_2, u_3, \,..\, u_n$. We will take the repayments of capital.

Referring to Art. (51), we see that the *present value* of the repayments of capital is

$$\begin{aligned}
&u_1v-ivX\\
&+\{u_2v^2-ivX+iu_1v^2\}\\
&+\{u_3v^3-ivX+iu_1v^2+iu_2v^3\}\\
&+\{u_4v^4-ivX+iu_1v^2+iu_2v^3+iu_3v^4\}\\
&+\;\cdots\cdots\\
&+\{u_nv^n-ivX+iu_1v^2+iu_2v^3+iu_3v^4+\,..\,+iu_{n-2}v^{n-1}+iu_{n-1}v^n\}
\end{aligned}$$

* This subject will be found fully discussed in Arts. 67 and following.

This sum becomes

$$\begin{aligned}
&X-nivX+niv(u_1v+u_2v^2+\;..\;+u_{n-1}v^{n-1})-iv\{u_1v+2u_2v^2+\;..\\
&\qquad\qquad\qquad\qquad\qquad\qquad+(n-1)u_{n-1}v^{n-1}\}\\
&=X-nivX+niv(X-u_nv^n)-iv\{u_1v+2u_2v^2+\;..\\
&\qquad\qquad\qquad\qquad\qquad\qquad+(n-1)u_{n-1}v^{n-1}\}\\
&=X-iv\{u_1v+2u_2v^2+\;..\;+(n-1)u_{n-1}v^{n-1}+nu_nv^n\}\\
&=X-iv^2\{u_1+2u_2v+\;..\;+(n-1)u_{n-1}v^{n-2}+nu_nv^{n-1}\}.
\end{aligned}$$

The second term with the negative sign is of course the present value of the payments for interest. This symmetrical expression will be considered hereafter, in the application of the higher mathematics to the theory of compound interest (see Chapter VI.).

NOTE.—It is to be understood that no attempt has been made to treat the subject of Varying Annuities in an exhaustive manner; but it is thought that sufficient has been given to prove of service to those desiring to continue the investigation of the subject for themselves.

CHAPTER IV.

On the Determination of the Rate of Interest, where the Amount of Capital repaid is the same as that advanced.

NOTE.—Throughout, where a dash is attached to a symbol, no distinction is implied by the direction of the dash. Thus $a' = a'$, $i' = i'$, and so on.

(54) When the periodic payments of an annuity are given, and their value known, the question arises for determination of the rate of interest that must be assumed in order to furnish the given value of the annuity. This question is one of practical importance, and its determination not free from difficulty. **Statement of the problem.**

Referring to Chapter II., Art. (23), we know that

$$ia_{\overline{n}|} + v^n = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is, that a sum 1 will yield an annuity of i for n years, and at the same time provide for the repayment of 1 at the end of n years, no matter what value is given to i; in other words, no matter what rate of interest capital is assumed to bear for the time being.

When, therefore, $a_{\overline{n}|}$ is given, we have for the determination of the corresponding value of i, the equation

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

$$= \frac{1-(1+i)^{-n}}{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From this equation the value of i cannot be exactly ascertained, but by the aid of various methods of approximation it can be determined with as much accuracy as may be required.

Let us now write x for i, and a for $a_{\overline{n}|}$ in (2), so that we have to determine x from the equation

$$a=\frac{1-(1+x)^{-n}}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

On the formula for giving a first approximation to the value required.

(55) Suppose that we have found, by reference to tables or otherwise, that x cannot differ much, say, from a rate i; so that $x=i+\rho$ (say) where ρ is relatively small compared with x and i, and positive or negative as the case may be.

Then we have $$a=\frac{1-(1+i+\rho)^{-n}}{i+\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and expanding $(1+i+\rho)^{-n}$ by the binomial theorem, we get

$$ai+a\rho=1-\left\{(1+i)^{-n}-n(1+i)^{-(n+1)}\rho+\frac{n\,.\,n+1}{2}(1+i)^{-(n+2)}\rho^2-\&c.\right\} \quad . \quad . \quad (5)$$

As ρ is by the conditions relatively small, we have, neglecting all terms of the expansion except the first two,

$$ai+a\rho=1-\{(1+i)^{-n}-n(1+i)^{-(n+1)}\rho\}\ ;$$

$$\therefore \quad \rho\{a-n(1+i)^{-(n+1)}\}=1-(1+i)^{-n}-ai$$

$$=i\left(\frac{1-(1+i)^{-n}}{i}-a\right).$$

If we write v for $(1+i)^{-1}$, and call $\frac{1-(1+i)^{-n}}{i}=\frac{1-v^n}{i}=a'$, the above becomes

$$\rho(a-nv^{n+1})=i(a'-a)\ ;$$

$$\therefore \quad \rho=\frac{i(a'-a)}{a-nv^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad (A)$$

and $$x=i+\rho\,;$$

$$\therefore \quad x=i\left(1+\frac{a'-a}{a-nv^{n+1}}\right).$$

On the determination of the formula for giving a second and nearer approximation to the value required.

(56) If, in the expansion of $(1+i+\rho)^{-n}$, we retain the term involving the second power of ρ, we have for the determination of ρ the quadratic equation in ρ,

$$ai+a\rho=1-\left\{(1+i)^{-n}-n(1+i)^{-(n+1)}\rho+\frac{n\,.\,n+1}{2}(1+i)^{-(n+2)}\rho^2\right\} \quad . \quad (6)$$

The value of ρ, given by the solution of this equation in the ordinary way, would be cumbrous, and troublesome for the purposes of calculation; and the following method will be found to give the approximate value of ρ readily.

Since ρ is relatively small, the value of $\frac{n \,.\, n+1}{2}(1+i)^{-(n+2)}\rho^2$ will not be much affected by writing, for ρ^2, $\rho \times \frac{i(a'-a)}{a-nv^{n+1}}$; the second factor, $\frac{i(a'-a)}{a-nv^{n+1}}$, being the value of ρ given by (A), and obtained on the assumption that the term involving ρ^2 may be neglected.

Thus (6) becomes reduced to

$$ai+a\rho=1-\left\{v^n-nv^{n+1}\rho+\frac{n \,.\, n+1}{2}v^{n+2}\frac{i(a'-a)}{a-nv^{n+1}}\rho\right\},$$

which gives

$$\rho=\frac{i(a'-a)}{a-nv^{n+1}+\frac{n \,.\, n+1}{2}v^{n+2}\frac{i(a'-a)}{a-nv^{n+1}}} \quad . \quad . \quad . \quad . \quad (B)$$

(57) We may proceed somewhat differently, and more correctly, as follows:—

Another and more correct method of approximation.

Reverting to the expression given in (4), we have

$$a=\frac{1-(1+i+\rho)^{-n}}{i+\rho}$$

$$=\{1-(1+i+\rho)^{-n}\}(i+\rho)^{-1}$$

$$=\{1-(1+i+\rho)^{-n}\}\left(1+\frac{\rho}{i}\right)^{-1}\frac{1}{i}.$$

Expanding, as before, by the binomial theorem, we have

$$a=\left\{1-(1+i)^{-n}+n \,.\, (1+i)^{-(n+1)}\rho-\frac{n \,.\, n+1}{2}(1+i)^{-(n+2)}\rho^2+\&c.\right\}$$
$$\times\left(\frac{1}{i}-\frac{\rho}{i^2}+\frac{\rho^2}{i^3}-\&c.\right)$$

$$=\left\{1-v^n+nv^{n+1}\rho-\frac{n \,.\, n+1}{2}v^{n+2}\rho^2+\&c.\right\}\times\left\{\frac{1}{i}-\frac{\rho}{i^2}+\frac{\rho^2}{i^3}-\&c.\right\}$$

$$=\frac{1-v^n}{i}+\frac{nv^{n+1}}{i}\rho-\frac{1-v^n}{i^2}\rho-\frac{n \,.\, n+1}{2}\frac{v^{n+2}}{i}\rho^2-\frac{nv^{n+1}}{i^2}\rho^2$$
$$+\frac{1-v^n}{i^3}\rho^2-\&c. \quad . \quad . \quad . \quad . \quad (7)$$

In this expansion (7), we have only written down terms involving no higher powers of ρ than the second.

First approximation.

If now in (7) the terms involving ρ^2 be neglected, we get

$$a=\frac{1-v^n}{i}+\frac{nv^{n+1}}{i}\rho-\frac{1-v^n}{i^2}\rho$$

$$=a'+\frac{nv^{n+1}}{i}\rho-\frac{a'}{i}\rho,$$

from which we get

$$\rho=\frac{i(a'-a)}{a'-nv^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(C)}$$

Second approximation.

(58) If, now, proceeding in exactly the same way as in obtaining (B), we write, for ρ^2, $\rho\times\frac{i(a'-a)}{a'-nv^{n+1}}$, then, substituting in (7), we get

$$a=a'+\frac{nv^{n+1}}{i}\rho-\frac{a'}{i}\rho-\frac{n.n+1}{2}\frac{v^{n+2}}{i}\frac{i(a'-a)}{a'-nv^{n+1}}\rho-\frac{nv^{n+1}}{i^2}\frac{i(a'-a)}{a'-nv^{n+1}}\rho$$
$$+\frac{a'}{i^2}\frac{i(a'-a)}{a'-nv^{n+1}}\rho$$

$$a=a'+\frac{nv^{n+1}}{i}\left\{1-\frac{a'-a}{a'-nv^{n+1}}\right\}\rho-\frac{a'}{i}\left\{1-\frac{a'-a}{a'-nv^{n+1}}\right\}\rho$$
$$-\frac{n.n+1}{2}v^{n+2}\frac{a'-a}{a'-nv^{n+1}}\rho\,;$$

$$\therefore\quad \rho\left\{\left(1-\frac{a'-a}{a'-nv^{n+1}}\right)\frac{a'}{i}-\frac{nv^{n+1}}{i}\left(1-\frac{a'-a}{a'-nv^{n+1}}\right)\right.$$
$$\left.+\frac{n.n+1}{2}v^{n+2}\frac{a'-a}{a'-nv^{n+1}}\right\}=a'-a$$

$$\therefore\quad \rho=\frac{i(a'-a)}{\left(1-\frac{a'-a}{a'-nv^{n+1}}\right)(a'-nv^{n+1})+\frac{n.n+1}{2}v^{n+2}\frac{i(a'-a)}{a'-nv^{n+1}}}$$

$$=\frac{i(a'-a)}{\frac{a-nv^{n+1}}{a'-nv^{n+1}}\cdot(a'-nv^{n+1})+\frac{n.n+1}{2}v^{n+2}\frac{i(a'-a)}{a'-nv^{n+1}}}$$

$$=\frac{i(a'-a)}{a-nv^{n+1}+\frac{n.n+1}{2}v^{n+2}\frac{i(a'-a)}{a'-nv^{n+1}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(D)}$$

(59) It will be noticed that the formulas (A) and (C) differ in the denominator, the former having $a-nv^{n+1}$ for its denominator, and the latter $a'-nv^{n+1}$.

Comparison of results obtained.

As regards the formulas (B) and (D), it will be seen that they differ only in the second factor of the second term of the denominator, the former having $a-nv^{n+1}$ where the latter has $a'-nv^{n+1}$. The formula (D) may be taken as sufficiently near for all practical purposes where $a'-a$ is small—that is, where the approximate value i is itself near to the true value $i+\rho$.

It may be pointed out that formula (D) would be obtained if $\rho^2=\dfrac{i(a'-a)}{a'-nv^{n+1}}\rho$ were substituted in (6) instead of (7).

(60) The late Professor De Morgan gave a formula for a close approximation in vol. viii., p. 67, of the *Journal of the Institute of Actuaries*.

Another and still nearer approximation, given by De Morgan.

This formula is as follows:—

Let
$$p=(a'-a)i \qquad s=k+\frac{n+2}{3}q$$
$$q=a-nv^{n+1}$$
$$k=n.\overline{n+1}v^{n+1} \qquad t=\frac{vp}{q^2},$$

then
$$\rho=\frac{p}{q}\left\{1-\frac{kt}{2}(1-st)\right\}.$$

This formula gives very nearly indeed the correct value of ρ. In the example given by De Morgan, where $n=100$, $a=9{\cdot}5233704$, the result taking $i={\cdot}10$ is $i+\rho$ or the actual rate $={\cdot}10499991$, while it should be ·105—an approximation certainly sufficiently near for ordinary purposes.

(61) This formula is not demonstrated by De Morgan; but may be thus obtained:—

Demonstration of the formula given by De Morgan.

From
$$a=\frac{1-(1+i+\rho)^{-n}}{i+\rho}$$

we have, using similar methods to that of Art. (56),

$$\frac{n.\overline{n+1}.\overline{n+2}}{2.3}v^{n+3}\rho^3-\frac{n.n+1}{2}v^{n+2}\rho^2-(a-nv^{n+1})\rho+(a'-a)i=0\ ;$$

or, using the above notation,

$$\frac{k}{2}\frac{(s-k)}{q}v^2\rho^3-\frac{k}{2}v\rho^2-q\rho+p=0.$$

Putting, as before,

$$\rho^2=\rho\times\frac{p}{q},$$

and

$$\rho^3=\rho\times\frac{p^2}{q^2},$$

the equation becomes

$$\frac{k}{2}\frac{(s-k)}{q}v^2\rho\times\frac{p^2}{q^2}-\frac{k}{2}v\rho\times\frac{p}{q}-q\rho+p=0\,;$$

or

$$\rho\left\{q+\frac{k}{2}\times\frac{pv}{q}-\frac{k}{2}(s-k)qt^2\right\}=p\,;$$

or

$$\rho=\frac{p}{q+\frac{k}{2}\left(\frac{pv}{q}-(s-k)qt^2\right)}$$

$$=\frac{\frac{p}{q}}{1+\frac{k}{2}\left\{t-(s-k)t^2\right\}}$$

$$=\frac{p}{q}\left\{1-\frac{kt}{2}(1-st)\right\}\text{ approximately.}$$

On a working formula involving finite differences only.

(62) The formulas (B), (D), and De Morgan's formula Art. (60), require a considerable amount of calculation. Two other working formulas have recently been given by Mr. G. F. Hardy where the quantities involved are the finite differences only.

Let a be the value of an annuity, at a rate of interest i (which it is required to find), and let a_1, a_2, a_3, be the tabulated annuity-values for the same term at the rates i_1, i_1+h, i_1+2h, respectively, where a lies between a_1 and a_2.

Then by the ordinary theorem of finite differences we have, the interval for differencing being h,

$$a=a_1+\frac{\rho}{h}\Delta a_1+\frac{\frac{\rho}{h}\cdot\frac{\rho}{h}-1}{1.2.}\Delta^2a_1+\&c.$$

$$=a_1+\frac{\rho}{h}(\Delta a_1-\tfrac{1}{2}\Delta^2a_1)+\frac{\rho^2}{h^2}\frac{\Delta^2a_1}{2}\quad . \quad . \quad . \quad . \quad (1)$$

neglecting all differences beyond the first two.

If now we proceed exactly in the same way as in obtaining (B) and (D), and write for $\frac{\rho^2}{h^2}$, the value $\frac{\rho^2}{h^2}=\frac{\rho}{h}\times\frac{a-a_1}{\Delta a_1-\frac{1}{2}\Delta^2 a_1}$, where $\frac{a-a_1}{\Delta a_1-\frac{1}{2}\Delta^2 a_1}$ is the value obtained for $\frac{\rho}{h}$ in (1), if we neglect the term $\frac{\rho^2}{h^2}\frac{\Delta^2 a_1}{2}$, we get

$$a=a_1+\frac{\rho}{h}(\Delta a_1-\tfrac{1}{2}\Delta^2 a_1)+\frac{\rho}{h}\cdot\frac{a-a_1}{\Delta a_1-\frac{1}{2}\Delta^2 a_1}\frac{\Delta^2 a_1}{2};$$

$$\therefore\quad \frac{\rho}{h}=\frac{a-a_1}{\Delta a_1-\frac{1}{2}\Delta^2 a_1+\dfrac{a-a_1}{\Delta a_1-\frac{1}{2}\Delta^2 a_1}\cdot\dfrac{\Delta^2 a_1}{2}}\quad .\ .\ .\ .\ .\ .\quad (D_1)$$

This will be found to be a working formula giving results very nearly correct.

Let us assume $\Delta^3 a_1=k\Delta^2 a_1=k^2\Delta a_1$, where k is small,* then we have

$$1+k=(1+k)^{\frac{1}{2}}(1+k)^{\frac{1}{2}}$$

$$=\frac{(1+k)^{\frac{1}{2}}}{(1+k)^{-\frac{1}{2}}}$$

$$=\frac{1+\frac{k}{2}}{1-\frac{k}{2}}\ \text{approximately}\quad .\ .\ .\ .\ .\ .\ .\quad (a)$$

Also

$$1=\frac{1+\frac{k}{2}}{1+\frac{k}{2}}$$

$$=\left(1+\frac{k}{2}\right)\left(1+\frac{k}{2}\right)^{-1}$$

$$=\left(1+\frac{k}{2}\right)\left(1-\frac{k}{2}\right)\ \text{approximately}\quad .\ .\ .\quad (b)$$

* The assumption that the differences are in geometric progression, the constant ratio k being small, will be found on examination not to involve sensible error.

Hence, referring to (D_1), we have, remembering (a) and (b),

$$\frac{\rho}{h}=\frac{a-a_1}{\Delta a_1\left(1-\frac{k}{2}\right)+\frac{a-a_1}{\Delta a_1\left(1-\frac{k}{2}\right)}\cdot\frac{\Delta^2 a_1}{2}}$$

$$=\frac{(a-a_1)\Delta a_1\left(1+\frac{k}{2}\right)}{(\Delta a_1)^2\left(1-\frac{k}{2}\right)\left(1+\frac{k}{2}\right)+\frac{(a-a_1)\left(1+\frac{k}{2}\right)}{\left(1-\frac{k}{2}\right)}\frac{\Delta^2 a_1}{2}}$$

$$=\frac{\Delta a_1+\frac{1}{2}\Delta^2 a_1}{\frac{(\Delta a_1)^2}{a-a_1}+(1+k)\frac{\Delta^2 a_1}{2}}$$

$$=\frac{\Delta a_1+\frac{1}{2}\Delta^2 a_1}{\frac{(\Delta a_1)^2}{a-a_1}+\frac{\Delta^2 a_1}{2}}\text{ approximately } \quad . \quad . \quad . \quad . \quad . \qquad (D_2)$$

since the term $\frac{k\Delta^2 a_1}{2}=\frac{\Delta^3 a_1}{2}$ may be neglected.

This formula (D_2) is stated to give, on the whole, better results than (D_1).

Numerical illustrations of the formulas (D_1) and (D_2) will be found in Art. (71), and their connection with formula D will be found explained in Chapter VI.

In the event of the tables used giving the values of the reciprocals or of the logs of a_1, a_2, a_3, instead of the values a_1, a_2, a_3, themselves, it is stated no loss of accuracy would result from using either of the former instead of the latter.

On the determination of the rate of interest without the aid of annuity tables.

(63) In the paper by Professor De Morgan referred to in Art. (60), there is given an interesting *résumé* of the various attempts that have been made to obtain an approximate solution of the problem, which, as there pointed out, is one of some two centuries' standing. Before anything like extensive sets of annuity-values had been calculated and published, mathematicians appear to have confined their attention to obtaining a formula not involving the use of existing tables to get an approximate value.

Reference may also be made to Baily's *Doctrine of Interest and Annuities*, Baily, in this work, gives a method of his own which it will be sufficient to demonstrate here for the particular case of the present value of an annuity, the method itself being equally applicable to amounts of annuities and to deferred annuities.

Baily's formulas for the determination of the rate of interest for the values and amounts of annuities-certain, as well as for the values of deferred annuities, without the aid of tables.

We have

$$a=\frac{1-(1+i)^{-n}}{i}$$

$$=\frac{1-\left\{1-ni+\frac{n.\overline{n+1}}{2}i^2-\frac{n.\overline{n+1}.\overline{n+2}}{2.3}i^3+\&c.\right\}}{i}$$

$$=n-\frac{n.\overline{n+1}}{2}i+\frac{n.\overline{n+1}.\overline{n+2}}{2.3}i^2-\&c.$$

$$\therefore\quad \frac{a}{n}=1-\frac{n+1}{2}i+\frac{\overline{n+1}.\overline{n+2}}{2.3}i^2-\&c.$$

$$\therefore\quad \left(\frac{a}{n}\right)^{-\frac{2}{n+1}}=\left\{1-\frac{n+1}{2}i+\frac{\overline{n+1}.\overline{n+2}}{2.3}i^2-\&c.\right\}^{-\frac{2}{n+1}}.$$

Expanding the right-hand side of this equation by the multinomial theorem, we get (see *Algebra—Multinomial Theorem*)

$$\left(\frac{a}{n}\right)^{-\frac{2}{n+1}}=\left(\frac{n}{a}\right)^{\frac{2}{n+1}}=1+i-\frac{n-1}{12}i^2+\&c. \quad . \quad . \quad . \quad (1).$$

The coefficient of the term involving i^3 will be found to be zero.

Writing $\beta=\left(\frac{n}{a}\right)^{\frac{2}{n+1}}-1$, and neglecting higher powers of i,

we have $\qquad \beta=i-\frac{n-1}{12}i^2=i-pi^2$, say.

The first approximation to the value of i is $i=\beta$ (*see* Art. 56). The second approximation is given by putting $i^2=i\beta$, whence

$$\beta=i-pi\beta;$$

$$\therefore\quad i=\frac{\beta}{1-p\beta}.$$

The third approximation is found by putting $i^2=i\cdot\frac{\beta}{1-p\beta}$, whence

$$\beta=i-pi\frac{\beta}{1-p\beta};$$

$$\therefore\quad i=\frac{\beta}{1-\dfrac{p\beta}{1-p\beta}}$$

$$=\frac{\beta(1-p\beta)}{1-2p\beta}$$

$$=\frac{\{12-(n-1)\beta\}\beta}{12-2(n-1)\beta}\text{ where }\beta=\left(\frac{n}{a}\right)^{\frac{2}{n+1}}-1\quad .\quad .\quad (\mathrm{E}).$$

Similarly, if A be used to denote $\frac{(1+i)^n-1}{i}$, then the value of i is given by the formula

$$i=\frac{\{12+(n+1)\gamma\}\gamma}{12+2(n+1)\gamma},\text{ where }\gamma=\left(\frac{\mathrm{A}}{n}\right)^{\frac{2}{n-1}}-1\quad .\quad .\quad .\quad (\mathrm{F}).$$

Again, if B be used to denote the value of a deferred annuity for n years after m years, then $\mathrm{B}=\frac{(1+i)^{-m}-(1+i)^{-(m+n)}}{i}$, and the value of i is given by the formula

$$i=\frac{\{12(2m+n+1)-(n^2-1)\delta\}\delta}{12(2m+n+1)-2(n^2-1)\delta}\text{ where }\delta=\left(\frac{n}{\mathrm{B}}\right)^{\frac{2}{2m+n+1}}-1\quad .\quad (\mathrm{G}).$$

In Note D, p. 137, Baily gives a formula for another case, namely, where the annuity deferred is a perpetuity, that is, in formula G when $n=\infty$.

The formula of this nature, given by Professor De Morgan, on p. 65, vol. viii., of the *Journal of the Institute of Actuaries*, affords apparently a good approximation to the true value, but the process of computation is somewhat intricate.

Formulas by De Morgan, for the same purposes as Baily's.

De Morgan proceeds as follows:—Let i be the interest of 1 for one year, n the number of years of an annuity payable yearly, and a its present value. Given a and n, required i.

We have $$a=\frac{1-(1+i)^{-n}}{i}.$$

Let $$t=\frac{1}{3}\cdot\frac{n-1}{n+1}\,\frac{\log n-\log a}{\cdot 4342945},\text{ then}$$

$$\log(1+i)=\frac{\log n-\log a}{n+1}\left\{1+\frac{1}{1-t}-\frac{1}{20}\,\frac{t^3(1+t)}{(1-t)^2}\right\}\quad .\quad .\quad (\mathrm{E}_1)$$

Similarly for the amount of the annuity, if $A=\dfrac{(1+i)^n-1}{i}$,

then let $t=\dfrac{1}{3}\cdot\dfrac{n+1}{n-1}\cdot\dfrac{\log A-\log n}{\cdot 4342945}$, and we have

$$\log(1+i)=\frac{\log A-\log n}{n-1}\left\{1+\frac{1}{1+t}+\frac{1}{20}\frac{t^3(1-t)}{(1+t)^2}\right\}\quad . \quad . \quad (F_1)$$

Again, for the value of an annuity for n years deferred m years, if B denote its value, then

$$B=\frac{(1+i)^{-m}-(1+i)^{-(m+n)}}{i}.$$

Let $$t=\frac{1}{3}\cdot\frac{n^2-1}{(2m+n+1)^2}\cdot\frac{\log n-\log B}{\cdot 4342945},\text{ then}$$

$$*\log(1+i)=\frac{\log n-\log B}{2m+n+1}\cdot\left\{1+\frac{1}{1-t}-\frac{1}{20}t^3\right\}\text{ approximately }\ .\ (G_1)$$

No demonstration of these formulas is given by De Morgan, but in the Note appended to this Chapter will be found an indication of how the formulas may be deduced.

The following table shows the relative degree of accuracy obtained by the use of Baily's formula (E) and De Morgan's formula (E_1).

n	True Rate of Interest $i=\cdot 05$	
	Rate by Baily's formula (E).	Rate by De Morgan's formula (E_1).
10	·05000	·05000
25	·05003	·05001
50	·05019	·05004
75	·05060	·05014
100	·05140	·05019

At 10 per-cent ($i=\cdot 1$) and $n=21$, De Morgan's formula (E_1) gives $i=\cdot 1001$; and for $n=100$, the value given for i is $i=\cdot 1063$. De Morgan's formula (F_1) for amounts of annuities also gives results near the truth.

* De Morgan says that probably in this formula (G_1) $t^3\dfrac{(1+t)}{(1-t)^2}$ would be found to be more accurate than t^3, thus taking the same form as in formula (E_1).

ON THE DETERMINATION OF THE ACTUAL RATE OF INTEREST PAID BY A BORROWER, WHERE THE AMOUNT OF CAPITAL REPAID IS DIFFERENT FROM THE AMOUNT ADVANCED.

(See Note at beginning of this Chapter.)

On the actual rate of interest in the case of a loan to be repaid with an addition to the sum advanced.

(64) Once more reverting to Art. (54), let us suppose that the person who invests his capital does so on the terms that, although he is to receive the full benefit of interest at the rate i thereon for n years, together with repayment in full at the end of n years, yet the borrower is willing to take, instead of 1, the reduced sum of $1-p$, undertaking to pay interest on the full amount 1, and to repay that amount at the end of n years. It is evident that, under these circumstances, the lender is doing something more than investing his money at the rate i. As a matter of fact, he only advances $1-p$, and as he receives i as interest thereon, this is in itself equivalent to investing his money at the rate $\frac{1}{1-p}i$. In addition, instead of simply receiving back the sum of $1-p$ at the end of the n years, he actually receives 1—that is, the loan is repaid with a premium or bonus addition.

Now, since $$ia_{\overline{n}|}+v^n=1\ ;$$

$$\therefore\quad i(1-p)a_{\overline{n}|}+(1-p)v^n=1-p.$$

Let i' be such a rate of interest that

$$\left.\begin{array}{ll} (1-p)a_{\overline{n}|}=a'_{\overline{n}|} & \quad .\ .\ .\ (a) \\ \text{and}\qquad (1-p)v^n=v'^n & \quad .\ .\ .\ (b) \end{array}\right\}$$

where $$v'=(1+i')^{-1}\quad\text{and}\quad a'_{\overline{n}|}=\frac{1-v'^n}{i'}.$$

Now, since conditions (a) and (b) are to hold simultaneously we have, from (b),

$$v^n=\frac{v'^n}{1-p},$$

and substituting in (a), we get

$$(1-p)\frac{1-\dfrac{v'^n}{1-p}}{i}=a'_{\overline{n}|}\,;$$

$$\therefore\quad ia'_{\overline{n}|}+v'^n=1-p \quad . \quad . \quad . \quad . \quad (1)$$

We see that (1) indicates that if the lender only advances $1-p$, but the debt bearing interest at the rate i is considered to be 1, to be repaid in full at the end of n years, this is practically the same thing as the lender investing the sum of $1-p$ at a higher rate of interest than that nominally paid by the lender.

From (1) we have at once,

$$p=1-v'^n-ia'_{\overline{n}|}$$

$$=\frac{1-v'^n}{i'}(i'-i) \quad . \quad . \quad . \quad . \quad . \quad (E)$$

$$=a'_{\overline{n}|}(i'-i)\,;$$

that is, the loan or debt bearing interest at i, to be repaid at the end of n years, being 1, and the sum actually advanced by the lender being $1-p$, the difference between the two, or "the discount on the loan," namely p, is equivalent to the value of an annuity for the term calculated at the rate of interest actually paid by the borrower of the difference between the rates of interest actually and nominally paid.

The value of i', the rate of interest actually paid and received, will clearly depend upon the values of p and n.

The formula (E) may be put into the form

$$p=\{i'(1-p)-i\}\frac{(1+i')^n-1}{i'} \quad . \quad . \quad . \quad . \quad (E_1)$$

Here $i'(1-p)$ represents the interest on the capital advanced, $1-p$, at the rate actually made by the lender; and the difference between this and i, the interest actually received by him, accumulated for n years, will amount to the additional sum p which he is then to receive over and above the $1-p$ advanced. (See Example 6 of this Chapter in Chapter V.)

Further consideration of the subject of Art. (64).

(65) Let us suppose a series of such obligations entered into, so that

a sum of $1-p_1$	is borrowed, to be repaid at end of		1	year, with an addition of		p_1
„ $1-p_2$	„	„	2	„	„	p_2
„ $1-p_3$	„	„	3	„	„	p_3
	&c.			&c.		
„ $1-p_{n-1}$	„	„	$n-1$	„	„	p_{n-1}
„ $1-p_n$	„	„	n	„	„	p_n.

It is evident that the rate of interest actually paid by the lender will vary with each transaction; and if the various obligations are incurred with different investors, each investor will receive an actual rate of interest different from the other investors.

The borrower may, however, be desirous to ascertain the actual rate of interest paid by him considering the transactions as a whole.

If i be, as before, the nominal rate of interest, and i' represent the actual rate of interest paid by the borrower, treating the transactions as a whole, we have, from (1),

$$\left.\begin{array}{l} ia'_{\overline{1}|} \quad +v' \\ +ia'_{\overline{2}|} \quad +v'^2 \\ +ia'_{\overline{3}|} \quad +v'^3 \\ \quad \&c. \\ +ia'_{\overline{n-1}|}+v'^{n-1} \\ +ia'_{\overline{n}|} \quad +v'^n \end{array}\right\} = \left\{\begin{array}{l} 1-p_1 \\ +1-p_2 \\ +1-p_3 \\ +\&c. \\ +1-p_{n-1} \\ +1-p_n \end{array}\right.$$

Hence, by summation, we have

$$i(a'_{\overline{1}|}+a'_{\overline{2}|}+\ldots+a'_{\overline{n}|})+v'+v'^2+v'^3+\ldots+v'^n=n-(p_1+p_2+\ldots+p_n);$$

$$\therefore \quad i\left(\frac{1-v'}{i'}+\frac{1-v'^2}{i'}+\ldots+\frac{1-v'^n}{i'}\right)+a'_{\overline{n}|}=n-(p_1+p_2+\ldots+p_n)$$

$$\therefore \quad i\left(\frac{n-v'-v'^2-v'^3-\ldots\ldots-v'^n}{i'}\right)+a'_{\overline{n}|}=n-(p_1+p_2+\ldots+p_n)$$

$$\therefore \quad \frac{i(n-a'_{\overline{n}|})}{i'}+a'_{\overline{n}|}=n-(p_1+p_2+\ldots+p_n);$$

$$\therefore \quad p_1+p_2+\ldots+p_n=\frac{n-a'_{\overline{n}|}}{i'}(i'-i) \quad \ldots\ldots \quad (F)$$

(66) Let us now take the most general case.

Consideration of the general case of Art. (64).

Let us suppose c_{n_1}, c_{n_2}, c_{n_3} . . . to be the amounts respectively repayable at the end of n_1, n_2, n_3 . . . years, bearing interest in the meantime at the nominal rate i; and let $c_{n_1}-p_1$, $c_{n_2}-p_2$, &c., respectively denote the sums actually advanced in respect of c_{n_1}, c_{n_2}, &c. Then we have, if i', as before, denote the actual rate of interest paid by the borrower, considering the various transactions as a whole,

$$i(c_{n_1}a'_{\overline{n_1}|}+c_{n_2}a'_{\overline{n_2}|}+c_{n_3}a'_{\overline{n_3}|}+\ldots\ldots)+c_{n_1}(1+i')^{-n_1}+c_{n_2}(1+i')^{-n_2}$$
$$+c_{n_3}(1+i')^{-n_3}+\ldots$$
$$=c_{n_1}-p_1+c_{n_2}-p_2+c_{n_3}-p_3+\ldots$$

$$\therefore \ \frac{i}{i'}\{c_{n_1}-c_{n_1}(1+i')^{-n_1}+c_{n_2}-c_{n_2}(1+i')^{-n_2}+c_{n_3}-c_{n_3}(1+i')^{-n_3}$$
$$+\ldots\ldots\}c_{n_1}(1+i')^{-n_1}+c_{n_2}(1+i')^{-n_2}+c_{n_3}(1+i')^{-n_3}+\text{\&c.}$$
$$=c_{n_1}+c_{n_2}+c_{n_3}+\ldots-(p_1+p_2+p_3+\ldots)$$

$$\therefore \ p_1+p_2+p_3+\ldots\ldots$$
$$=\left(1-\frac{i}{i'}\right)\{c_{n_1}+c_{n_2}+c_{n_3}+\ldots-c_{n_1}(1+i')^{-n_1}-c_{n_2}(1+i')^{-n_2}-\text{\&c.}\}$$
$$=\frac{i'-i}{i'}\{c_{n_1}+c_{n_2}+c_{n_3}+\ldots-c_{n_1}(1+i')^{-n_1}-c_{n_2}(1+i')^{-n_2}-\text{\&c.}\} \quad \ldots\ldots \quad (G)$$

On examination of the formula (E), where the capital to be repaid is 1, we see that $1-v'^n$ is evidently the difference between the capital receivable and its present value.

Similarly, in formula (F), n is the capital to be repaid, and $a'_{\overline{n}|}$ its present value; so that $n-a'_{\overline{n}|}$ is the difference between the capital receivable and its present value.

Again, in the general case of formula (G), we see that the capital receivable is $c_{n_1}+c_{n_2}+c_{n_3}+$&c., and its present value is $c_{n_1}(1+i')^{-n_1}+c_{n_2}(1+i')^{-n_2}+c_{n_3}(1+i')^{-n_3}+\ldots$; so that the expression which is multiplied by $\frac{i'-i}{i'}$ is the difference between the capital receivable and its present value.

(67) Hence we get the general result—

Generalization of the results of Arts. (64-66) relating to loan transactions.

$$\left.\begin{array}{c}\text{Difference between sum repaid}\\ \text{and sum advanced}\end{array}\right\}=\frac{i'-i}{i'}\left\{\begin{array}{c}\text{(Sum repaid less its}\\ \text{present value.)}\end{array}\right.$$

If we call C the capital receivable, A the sum advanced by the lender, C′ the present value of the capital receivable, then this result becomes

$$C-A=\frac{i'-i}{i'}(C-C') \quad . \quad . \quad . \quad . \quad . \quad (H)$$

If, therefore, in any given loan transaction the nominal rate of interest is known, and the rate of interest which the lender is to make on the transaction is also given, the value of C—A, that is, the discount on the loan, may be at once ascertained.

The formula (H) may be thus put:—

$$C'+(C-C')\frac{i}{i'}=A.$$

Now C′ is the present value of the capital receivable, and A is the present value of the entire loan (interest and capital). Hence $(C-C')\frac{i}{i'}$ denotes the present value of the interest receivable on the loan. (See Art. 53.)*

On the applicability of formula of Art. 67 to the determination of the rate of interest actually paid by the borrower.

(68) Let us now proceed to consider whether the same formula may be applied to determine the rate of interest actually paid by the borrower.

We have $C-C'=$ Capital receivable less its present value.

$$=\left\{\begin{array}{l}\text{Present value of interest receivable at}\\ \qquad\text{rate } i'.\end{array}\right.$$

$$\therefore \quad \frac{C-C'}{i'}=\left\{\begin{array}{l}\text{Present value of interest receivable per}\\ \qquad\text{each unit in the rate.}\end{array}\right.$$

$$\therefore \quad \frac{i'}{C-C'}=\left\{\begin{array}{l}\text{Rate of interest for which a payment of}\\ \qquad\text{1 down will provide;}\end{array}\right.$$

$$\therefore \quad \frac{i'(C-A)}{C-C'}=\left\{\begin{array}{l}\text{Rate of interest for which the discount,}\\ \qquad C-A\text{, will provide,}\end{array}\right.$$

$$= i'-i.$$

Hence, the difference between the actual rate of interest and the nominal rate of interest can be found approximately by assuming some value near to the true value of i', and substituting this value of i' in $\frac{i'(C-A)}{C-C'}$; and it is clear that the nearer this

* It is evident that if C – C′ be the present value of the interest when the interest received is at the rate i', then $(C-C')\frac{i}{i'}$ will be the present value when *the interest received* is at the rate i.

assumed value of i' is to the true value, the nearer will the exact value of $i'-i$ be approached.

(69) If we refer, now, to the formula (C) for the approximation to the actual rate of interest corresponding to a given value of an ordinary annuity for n years, namely, **Comparison with previously-obtained results.**

$$\rho=i'-i=\frac{i(a'-a)}{a'-nv^{n+1}} \text{ approximately};$$

then, comparing this with (H), we have $a'=C=$capital receivable; and instead of the approximate true rate i, and a' the corresponding value of the annuity, we have the actual rate i', and the actual value of the annuity A, that is, the sum advanced. And in the denominator we have C′, the value of the capital receivable at the actual rate i' instead of nv^{n+1}, which is the value of the capital receivable at the approximate true rate i. This will appear from the following:—

Referring to Chapter II., Art. (24), we know that

$$\left.\begin{array}{l}\text{Amount redeemed—that is, portion of capital}\\ \quad\text{receivable at end of } t\text{th year}\end{array}\right\}=v^{n-t+1}$$

$$\therefore \text{ Present value of do. do.} \qquad =v^{n-t+1}\times v^t$$

$$=v^{n+1}.$$

And, as there are n payments of annuity, we have

$$\text{Present value of capital}=v^n\times v+v^{n-1}\times v^2+\ldots+v^2\times v^{n-1}+v\times v^n$$

$$=nv^{n+1}.$$

It will be observed that, in the application of formula (H) to the determination of the exact rate of interest in any given loan transaction, the only quantity requiring calculation is the present value of the capital receivable; and, generally speaking, this will be found to be a simpler process than the computation of the present value of the periodic payments, constituted of interest and repayment of capital. (See Art. 53.)

Reference may be made to a paper by Mr. W. M. Makeham, p. 132 of vol. xviii. of the *Journal of the Institute of Actuaries*, where the subjects referred to in what precedes, are discussed at some length, and to Example (3) of the Illustrations of this Chapter in Chapter V.

Numerical illustrations of preceding Articles, Formulas A, B, C, and D.

(70) As a numerical illustration of formulas (A), (B), (C), (D), let us take an annuity for 100 years, whose present value is 9·5233704, required to ascertain the actual rate of interest.

If the annuity were to run for ever—that is, if it were a perpetuity—it is clear that the value would be somewhat greater than 9·5233704; that is, more nearly approaching to 10, which would be the value of a perpetuity at 10 per-cent.

Let us take 10 per-cent, or $i=10$, as the approximate value of i.

$$\log(1{\cdot}10)^{-100} = -4{\cdot}13927 = \bar{5}{\cdot}86073$$
$$= \log^{-1} {\cdot}00007257$$
$$\therefore \quad \frac{1-(1{\cdot}10)^{-100}}{{\cdot}10} = 9{\cdot}9992743 = a'$$
$$9{\cdot}5233704 = a$$
$${\cdot}4759039 = a'-a$$
$$9{\cdot}5167735$$
$$9{\cdot}9926774 = a'-nv^{n+1}$$

$$\log(1{\cdot}10)^{-101} = -4{\cdot}18066$$
$$= \bar{5}{\cdot}81934$$
$$\therefore \quad (1{\cdot}10)^{-101} = {\cdot}000065969 = v^{n+1}$$
$$\therefore \quad 100\times(1{\cdot}10)^{-101} = {\cdot}0065969 = nv^{n+1}$$
$$9{\cdot}5233704 = a$$
$$\therefore \quad 9{\cdot}5167735 = a-nv^{n+1}$$

Hence we have

Formula (A)—

$$\rho = \frac{{\cdot}10({\cdot}4759039)}{9{\cdot}5167735} = \frac{{\cdot}04759039}{9{\cdot}5167735} \quad \begin{array}{l} \log = \bar{2}{\cdot}6775192 \\ \log = 0{\cdot}9784897 \end{array}$$
$$(a) \quad \bar{3}{\cdot}6990295 = \log^{-1}{\cdot}00500068$$

Formula (C)—

$$\rho = \frac{{\cdot}10({\cdot}4759039)}{9{\cdot}9926774} = \frac{{\cdot}04759039}{9{\cdot}9926774} \quad \begin{array}{l} \log = \bar{2}{\cdot}6775192 \\ \log = 0{\cdot}9996818 \end{array}$$
$$(b) \quad \bar{3}{\cdot}6778374 = \log^{-1}{\cdot}004762527$$

$$(a)-(b) \quad {\cdot}0211921$$

Also

$$\frac{100\times101}{2} = 5050 = \frac{n\,.\,n+1}{2} \text{ and } \log(1{\cdot}10)^{-102} = -4{\cdot}18066 - {\cdot}04139$$
$$= -4{\cdot}22205$$
$$= \bar{5}{\cdot}77795$$

$$\begin{aligned}
&\text{Again, }\log 5050 &&= 3{\cdot}7032914 = \log\frac{n.\overline{n+1}}{2}\\
&\log(1{\cdot}10)^{-102} &&= \bar{5}{\cdot}77795 \quad = \log v^{n+2}\\
&\log &&= \bar{3}{\cdot}6990295 = \log\frac{i(a'-a)}{a-nv^{n+1}}\\
&\log{\cdot}0015145 &&= \bar{3}{\cdot}1802709 = \log\frac{n.\overline{n+1}}{2}v^{n+2}\frac{i(a'-a)}{a-nv^{n+1}}\\
& && \quad{\cdot}0211921\\
&\log{\cdot}0014424 &&= \bar{3}{\cdot}1590788 = \log\frac{n.\overline{n+1}}{2}v^{n+2}\frac{i(a'-a)}{a'-nv^{n+1}}
\end{aligned}$$

Hence we have,

Formula (B)—

$$=\frac{{\cdot}10({\cdot}4759039)}{9{\cdot}5167735+{\cdot}0015145}=\frac{{\cdot}04759039}{9{\cdot}5182880}\qquad \begin{aligned}\log&=\bar{2}{\cdot}6775192\\ \log&=0{\cdot}9785589\\ &\overline{\bar{3}{\cdot}6989603}\\ &=\log^{-1}{\cdot}00499989\end{aligned}$$

Formula (D)—

$$=\frac{{\cdot}10({\cdot}4759039)}{9{\cdot}5167735+{\cdot}0014424}=\frac{{\cdot}04759039}{9{\cdot}5182159}\qquad \begin{aligned}\log&=\bar{2}{\cdot}6775192\\ \log&=0{\cdot}9785556\\ &\overline{\bar{3}{\cdot}6989636}\\ &=\log^{-1}{\cdot}00499993\end{aligned}$$

The true value of ρ, as will appear from Art. (57), is $\rho={\cdot}005$; so that formulas (A), (B), and (D), give very close approximations to the exact value, ·10+·005, or ·105, whereas (C) is considerably in error. This arises from the fact that, taking 10 per-cent as the approximate rate, is a rougher first approximation than would generally be the case when tables are used.

Numerical Illustrations continued—Formulas D_1 and D_2.

(71) As numerical illustrations of formulas (D_1) and (D_2) let us take the following:—

Formula (D_1) is $$\frac{\rho}{h}=\frac{a-a_1}{\Delta a_1-\frac{1}{2}\Delta^2 a_1+\dfrac{a-a_1}{\Delta a_1-\frac{1}{2}\Delta^2 a}\cdot\dfrac{\Delta^2 a_1}{2}}\quad\text{(Art. 62).}$$

A. An annuity-certain for 30 years is bought for 19 years' purchase; what rate of interest is made on the investment?*

Here, using (D_1) and Interest Tables for every $\frac{1}{4}$ per-cent, we have

$$a=19{\cdot}0000;\ i={\cdot}03+\rho;\ h={\cdot}0025;$$

		Δ	Δ^2
at 3 per-cent	$a_1=19{\cdot}6004$	$-{\cdot}6185$	$+{\cdot}0286$
„ $3\frac{1}{4}$ „	$a_2=18{\cdot}9819$	$-{\cdot}5899$	
„ $3\frac{1}{2}$ „	$a_3=18{\cdot}3920$		

$$a-a_1=-{\cdot}6004;\ \tfrac{1}{2}\Delta^2={\cdot}0143;\ \Delta-\tfrac{1}{2}\Delta^2=-{\cdot}6328;$$

$$\therefore\ \rho={\cdot}0025\left(\frac{6328}{6004}-\frac{143}{6328}\right)^{-1}$$

$$={\cdot}0025(1{\cdot}054-{\cdot}023)^{-1}$$

$$=\frac{{\cdot}0025}{1{\cdot}031}={\cdot}002425;$$

whence $i={\cdot}032425,$

which is correct to the last figure.

B. As another example, we will take the case where the purchase-money is $a=11{\cdot}99051$, and the term$=30$ years; to find i.

Taking the values of the annuities at $7\frac{1}{4}$, $7\frac{1}{2}$, and $7\frac{3}{4}$ per-cent, from the Tables, we have

$$a=11{\cdot}99051$$

	Δ	Δ^2
$a_1=12{\cdot}10366$	$-{\cdot}29327$	$+{\cdot}01149$
$a_2=11{\cdot}81039$	$-{\cdot}28178$	
$a_3=11{\cdot}52861$		

$$h={\cdot}0025;\ a-a_1=-{\cdot}11315;\ \tfrac{1}{2}\Delta^2={\cdot}00575;\ \Delta-\tfrac{1}{2}\Delta^2=-{\cdot}29902;$$

$$\rho={\cdot}0025\left\{\frac{29902}{11315}-\frac{575}{29902}\right\}^{-1}.$$

whence ρ is found to be ${\cdot}000953$, and consequently

$$i=i_1+\rho={\cdot}072500+{\cdot}000953={\cdot}073453;$$

which, again, is correct to the last figure, the true value of i being ${\cdot}07345253$.

* This Article is taken from the paper by Mr. G. F. Hardy, in which the working formulas D_1 and D_2 are given.

If no tables are available which give annuity-values for every quarter per-cent, it will be found sufficient to use the logs or reciprocals of the annuity-values with the logs or reciprocals at intervals of $\frac{1}{2}$ per-cent. Thus, taking Example A, with the values of $\frac{1}{a}$ given in the Tables for 3, $3\frac{1}{2}$, and 4 per-cent, we have

$$\frac{1}{a}=\frac{1}{19}=\cdot 0526316$$

		Δ	Δ_2
$\frac{1}{a_1}$	$=\cdot 0510193$	$+\cdot 0033520$	$+\cdot 0001068$
$\frac{1}{a_2}$	$=\cdot 0543713$	$\cdot 0034588$	
$\frac{1}{a_3}$	$=\cdot 0578301$		

$h=\cdot 005$; $\frac{1}{a}-\frac{1}{a_1}=\cdot 0016123$; $\frac{1}{2}\Delta^2=\cdot 0000534$; $\Delta-\frac{1}{2}\Delta^2=\cdot 003296$;

whence

$$\rho=\cdot 005\left\{\frac{32986}{16123}+\frac{534}{32986}\right\}^{-1}$$

$$=\cdot 005(2\cdot 0621)^{-1}=\cdot 0024248$$

$$i_1=\cdot 0300000$$

$$\therefore\quad i=\cdot 0324248$$

The true value is $\cdot 032425206$,

which result differs from the truth by 4 in the 7th place. If the logs had been used instead of the reciprocals of the annuity-values, the result would have been in error by only 1 in the 7th place of decimals, or about $\frac{1}{100}$th of a penny per-cent. Even with intervals of 1 per-cent in the tabulated values, this method gives fair results. Thus in Example B, when the reciprocals are used for intervals of 1 per-cent, $i=\cdot 073450$, or by logs $i=\cdot 073455$, which results are respectively less and greater than the truth by the quantity $\cdot 0000025$; that is, there would be an error in the rate per-cent of $\frac{1}{16}$th of a penny.

If we apply the formula to the example used in Art. (70) in illustration of formulas (A), (B), (C), and (D), namely, $a=9\cdot 5233704$; term $=100$ years; and further assume that we

have only the values tabulated as far as 10 per-cent (taking the values at 8, 9, and 10 per-cent, and working with the reciprocals), it will be found that $i = \cdot 10499992$, a result almost identical with that obtained by formula (D) and De Morgan's formula Art. (60).

The formula may obviously be employed in the solution of any similar inverse problem; for example, to find the rate of interest returned to the purchaser of a Debenture Bond. In this case we have to solve the equation

$$i_1 - i - \frac{\rho}{a} = 0 \quad \text{(see formula (E), Art. (64))},$$

where i_1 is the nominal rate of interest, i the true rate, ρ the premium paid for the bond (a negative quantity if bought at a discount), and a the value of an annuity for the given term of n years at the rate i. Thus, let $i_1 = \cdot 06$; $\rho = \cdot 075$; $n = 20$ (the number of years at the end of which the bond is redeemable at par).

We have,

$$\text{at 4 per-cent } (i_1 - i) - \frac{\rho}{a} = \cdot 02 - \cdot 075\,(\cdot 07358175) = + \cdot 01448137 = u_1$$

$$\text{,, 5 ,,} \qquad = \cdot 01 - \cdot 075\,(\cdot 08024259) = + \cdot 00398181 = u_2$$

$$\text{,, 6 ,,} \qquad = \quad - \cdot 075\,(\cdot 08718456) = - \cdot 00653884 = u_3$$

Here $i = \cdot 04 + \rho$; $u - u_1 = - \cdot 01448137$; $\frac{1}{2}\Delta^2 = - \cdot 00001055$; $\Delta - \frac{1}{2}\Delta^2 = - \cdot 01048901$;

$$\therefore \quad \rho = \cdot 01 \left(\frac{1048901}{1448137} + \frac{1055}{1048901} \right)^{-1}$$

$$= \cdot 01378710;$$

whence $\quad i = \cdot 05378710$, a result which differs from the truth by unity in the 8th place.

Generally speaking, however, it is sufficient in these problems to use the ordinary interpolation by first differences; thus we get, in the above example, from u_2 and u_3,

$$i = \cdot 053785,$$

which is true to the 5th place. With rates at intervals of a quarter per-cent, the results by this latter method are, perhaps,

as exact as can be required for any purpose. Thus, taking the values of $\frac{1}{a}$ for $5\frac{1}{4}$ and $5\frac{1}{2}$ per-cent in the above example, we find $i = \cdot05378690$, which, though not so exact as the result given by the formula (D_1), yet differs by only $\frac{1}{200}$th of a penny per-cent from the true rate.

Formula (D_2) is $\quad \dfrac{\rho}{h} = \dfrac{\Delta a_1 + \frac{1}{2}\Delta^2 a_1}{\dfrac{(\Delta a_1)^2}{a - a_1} + \dfrac{\Delta^2 a_1}{2}}.$

If we apply this formula to Example A, we get (taking the values to an extra figure)

$$\Delta = -\cdot61852,\ \tfrac{1}{2}\Delta^2 = \cdot01432,$$

$$a - a_1 = -\cdot60044\,;$$

whence $\quad \rho = \cdot0025 \left\{ \dfrac{\cdot60420}{\dfrac{(\cdot61852)^2}{\cdot60044} - \cdot01432} \right\}$

$\log \cdot61852 = 9\cdot79135$	$\log \cdot60420 = 9\cdot78118$
	$-\log \cdot62283 = 9\cdot79437$
$\log (\cdot61852)^2 = 9\cdot58271$	$9\cdot98681$
$\log \cdot60044 = 9\cdot77847$	$+\log \cdot0025 = 7\cdot39794$
$9\cdot80424 = \log \cdot63715$	$\log \cdot0024252 = 7\cdot38475$
$-\ 1432$	
$\cdot62283$	$\therefore\ \rho = \cdot0024252$
	and $i = \cdot0324252$

which is true to the last figure.

In the same way we get for the value of i, in Example B, $i = \cdot0734528$ with an error of 3 in the last place.

Numerical illustrations continued—Formula H.

(72) In this Article it is proposed to give numerical illustrations of the general formula (H) of Art. (67), due to Mr. Makeham. We will, in this place, take examples of its direct application, that is, where the nominal rate of interest is known, and the rate of interest which the lender is to make on the transaction is also given, to find the discount on the loan; in other words, given C the capital to be repaid, i the nominal rate of interest, and i' the actual rate, to find A the amount to be advanced.

Example 1.—A sum of £125 is to be repaid as follows:—

One twenty-seventh at the expiration of 4 years.

" " " 5 "

And so, finally,

One twenty-seventh at the expiration of 30 years,

interest being payable in the meantime at the rate of 6 per-cent, required the value of the loan to yield the purchaser interest at the rate i'.

The payments of capital evidently consist of an annuity of $\frac{125}{27}$ for 27 years, first payment at the end of 4 years. The value of the capital receivable is therefore

$$C' = \frac{125}{27}(1+i')^{-3} \cdot \frac{1-(1+i')^{-27}}{i'}$$

$$C = \text{capital receivable} = 125.$$

Hence the value of A—that is, the value of the loan or the sum advanced—is given by

$$A = C' + (C - C')\frac{i}{i'} \qquad \text{(Art. 67)}$$

$$= \frac{125}{27}(1+i')^{-3} \cdot \frac{1-(1+i')^{-27}}{i'} + \left\{125 - \frac{125}{27}(1+i')^{-3} \times \frac{1-(1+i')^{-27}}{i'}\right\}\frac{\cdot 06}{i'}$$

$$= 125\left\{\frac{\cdot 06}{i'} + \left(1 - \frac{\cdot 06}{i'}\right)\frac{(1+i')^{-3}}{27} \cdot \frac{1-(1+i')^{-27}}{i'}\right\}.$$

When i' is known, the numerical calculation can be readily performed.

Example 2.—A loan of £10,000 is repayable (as follows) in 40 years, with a premium of 25 per-cent, viz.,

£ 94	(plus 25 per-cent)	at the end of	first	year,
102	"	"	second	"
110	"	"	third	"

and interest at the rate of 2·4 per-cent is payable in the meantime on the capital outstanding—that is, on the nominal amount of the

loan together with the premium addition of 25 per-cent. Required the value of the loan to yield interest to the lender at the rate of 5 per-cent.

Using Mr. Makeham's formula, Art. (45), we have

$$\text{Value of capital advanced} = 94\overset{1}{V}_{40} + 8\overset{2}{V}_{40}$$

$$= 94\overset{1}{V}_{40} + 8\,\frac{\overset{1}{V}_{40} - 40(1+i)^{-40}}{i} \quad \text{Art. (45).}$$

$$= \overset{1}{V}_{40}\left\{94 + \left(40 + \frac{1}{i}\right)\times 8\right\} - \frac{40\times 8}{i}$$

Now $\overset{1}{V}_{40}(i = \cdot 05) = 17{\cdot}15909$, and the above expression becomes

$$\text{Value of capital advanced} = 17{\cdot}15909 \times \{94 + 320 + 160\} - 6400$$
$$= 17{\cdot}15909 \times 574 - 6400$$
$$= 9849{\cdot}317 - 6400$$
$$= 3449{\cdot}317$$

$$\text{Value of interest receivable} = (10{,}000 - 3449{\cdot}317)\,\frac{2{\cdot}4}{5} \quad \text{(Art. 67)}$$
$$= 3144{\cdot}328$$

$$\left.\begin{array}{l}\text{Hence, value of capital advanced + value}\\ \qquad\text{of interest receivable}\end{array}\right\} = 3449{\cdot}317 + 3144{\cdot}328$$
$$= 6593{\cdot}645\,;$$

and as the capital advanced is to be returned with 25 per-cent addition, we have

$$\text{Value of loan} = 6593{\cdot}695 \times 1{\cdot}25$$
$$= 8242{\cdot}056.$$

It will be seen that this example is identical with that given in Arts. (46) and (48); for the annual payments in respect of interest and capital are

$$\text{1st year} \quad . \quad 1{\cdot}25 \times 94 + \cdot 024 \times 12{,}500 \; = 417{\cdot}5$$
$$\text{2nd} \;\;,, \quad . \quad 1{\cdot}25 \times 102 + \cdot 024 \times 12{,}382{\cdot}5 = 424{\cdot}68$$

and so on.

For further illustration of the formula (H) of Mr. Makeham, and of Mr. Gray's formula, Art. (47), see Example 2 of Chapters III. and IV., in Chapter V.

NOTE TO ART. (63).

On the Method of deducing De Morgan's Formulas (E_1), (F_1), (G_1).

Referring to equation (1) of Art. (63), we have

$$\frac{a}{n}^{-\frac{2}{n+1}} = 1 + i - \frac{n-1}{12} i^2 + \&c.$$

If we continue the series on the right-hand side, by the aid of the Multinomial Theorem, we get, uniting for $\frac{a}{n}^{-\frac{2}{n+1}}$ its equivalent $\frac{n}{a}^{\frac{2}{n+1}}$,

$$\frac{n}{a}^{\frac{2}{n+1}} = 1 + i - \frac{n-1}{12} i^2 + \frac{(n-1)(n+2)(n+3)}{1440} i^4 + \quad . \quad . \quad . \qquad (2)$$

We may write the right-hand side approximately in the form

$$(1+i)^{1 - \frac{n-1}{12} i + \frac{(n-1)(n+2)(n+3)}{1440} i^3 +}$$

$$= (1+i)^{\frac{1}{2}} \times (1+i)^{\frac{1}{2}\left\{1 - \frac{n-1}{6} i + \frac{(n-1)(n+2)(n+3)}{720} i^3 + \right\}} \quad . \quad . \quad . \quad . \qquad (3)$$

Now, referring to (2), and neglecting i^2 and higher powers, we get

$$\frac{n}{a}^{\frac{1}{n+1}} = (1+i)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

$$\therefore \quad \frac{1}{n+1} \log_{\epsilon} \frac{n}{a} = \log_{\epsilon} (1+i)^{\frac{1}{2}} = \frac{i}{2} \text{ approximately;}$$

$$\therefore \quad \frac{1}{3} \cdot \frac{n-1}{n+1} \log_{\epsilon} \frac{n}{a} = \frac{(n-1)i}{6} = t.$$

Hence
$$(1+i)^{\frac{1}{2}\left(1 - \frac{n-1}{6} i\right)} = (1+i)^{\frac{1}{2}(1-t)},$$

and since this is approximately equal to $\frac{n}{a}^{\frac{1}{n+1}}$, we have

$$(1+i)^{\frac{1}{2}(1-t)} = \frac{n}{a}^{\frac{1}{n+1}}$$

$$\therefore \quad (1+i)^{\frac{1}{2}} = \frac{n}{a}^{\frac{1}{n+1} \cdot \frac{1}{1-t}} \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

Hence, combining (a) and (b), we shall have

$$1 + i = (1+i)^{\frac{1}{2}} \times (1+i)^{\frac{1}{2}}$$

$$= \frac{n}{a}^{\frac{1}{n+1}} \times \frac{n}{a}^{\frac{1}{n+1} \cdot \frac{1}{1-t}}$$

$$= \frac{n}{a}^{\frac{1}{n+1}\left(1 + \frac{1}{1-t}\right)}$$

It is not clear what is the exact nature of the method adopted by De Morgan, in deducing the next term of the series forming the index of $\frac{n}{a}$ in formula (E_1).

Similarly, if $$A = \frac{(1+i)^n - 1}{i}, \text{ we have}$$

$$A = n + \frac{n.\overline{n-1}}{2} i + \ldots$$

$$\therefore \quad \frac{A}{n} = 1 + \frac{n-1}{2} i + \ldots$$

$$\therefore \quad \left(\frac{A}{n}\right)^{\frac{2}{n-1}} = 1 + i + \frac{n+1}{12} i^2 - \frac{\overline{n+1}.\overline{n-2}.\overline{n-3}}{1440} i^4.$$

Adopting precisely the same method as that just used, we get approximately

$$\left(\frac{A}{n}\right)^{\frac{1}{n-1}} = (1+i)^{\frac{1}{2}};$$

$$\therefore \quad \frac{1}{n-1} \log_\epsilon \frac{A}{n} = \frac{i}{2} \text{ approximately};$$

$$\therefore \quad \frac{1}{3} \frac{n+1}{n-1} \log_\epsilon \frac{A}{n} = \frac{n+1}{6} i = t;$$

$$\therefore \quad (1+i)^{\frac{1}{2}(1+t)} = \frac{A}{n}^{\frac{1}{n-1}};$$

$$\therefore \quad (1+i)^{\frac{1}{2}} = \frac{A}{n}^{\frac{1}{n-1} \cdot \frac{1}{1-t}};$$

and $$(1+i)^{\frac{1}{2}} = \frac{A}{n}^{\frac{1}{n-1}};$$

$$\therefore \quad 1 + i = \frac{A}{n}^{\frac{1}{n-1}\left(1 + \frac{1}{1+t}\right)}.$$

Here, again, it is not clear what is the exact nature of the method used by De Morgan in obtaining the next term of the series forming the index of $\frac{A}{n}$ in formula (F_1). It is evident that similar reasoning will apply in the third case, that of the formula (G_1).

CHAPTER V.

PRACTICAL EXAMPLES AND ILLUSTRATIONS.

The following practical examples and illustrations of the propositions contained in the preceding chapters, are here given.

CHAPTER I.

On accumulations in Consols.

(1).—A sum of £a is invested at the rate p per 1, and the dividends (3 per-cent, payable half-yearly) are invested in Consols as received at the rate of q per 1. To what amount of Consols will the whole have accumulated at the end of n years, allowing for income-tax at the rate of r per 1?

As p will purchase 1 of Consols,

„ a „ $\frac{a}{p}$ „

Now, the interest for half a year on 1 of Consols will be ·015,

and the income-tax payable thereon „ ·015r;

∴ the net amount of interest available for re-investment „ ·015$(1-r)$.

And this, invested in Consols at the rate q per 1, will yield an addition to Consols of $\frac{1}{q}(1-r)\cdot015$.

This is the net half-yearly rate of accumulation of the original amount of Consols, namely, $\frac{a}{p}$;

$\therefore$ Art. (9), the total accumulation in Consols at the end of n years will be $\frac{a}{p}\left\{1+\frac{1}{q}(1-r)\cdot 015\right\}^{2n}$.

Numerical Illustration.—A sum of £1,000 is invested at the rate $97\frac{1}{2}$ per 100 Consols, and the dividends are invested in Consols at the rate $98\frac{1}{2}$ per 100 Consols. To what amount of Consols will the whole have accumulated at the end of 15 years, allowing for income-tax at the rate of 6*d.* in the £ (*i.e.*, ·025 per 1)?

Here $a=1000$ $q=\cdot 985$ $2n=30$

$p=\cdot 975$ $r=\cdot 025$.

Hence

$$\text{Amount of Consols required}=\frac{1000}{\cdot 975}\left\{1+\frac{1}{\cdot 985}(1-\cdot 025)\cdot 015\right\}^{30}$$

$$=\frac{1000}{\cdot 975}\left\{1+\frac{\cdot 975}{\cdot 985}\times\cdot 015\right\}^{30}.$$

Now

$$\begin{aligned}\log\cdot 975&=\bar{1}\cdot 9890046\\ \log\cdot 015&=\bar{2}\cdot 1760913\\ \hline &\ \ \bar{2}\cdot 1650959\\ \log\cdot 985&=\bar{1}\cdot 9934362\\ \hline &\ \ \bar{2}\cdot 1716597\\ &=\log^{-1}01484772\end{aligned}$$

$$\begin{aligned}\log 1\cdot 01484772&=\cdot 0064010\\ \log(1\cdot 01484772)^{30}&=\cdot 192030\\ \log 1000&=3\cdot\\ \hline &\ \ 3\cdot 192030\\ \log\cdot 975&=\bar{1}\cdot 989005\\ \hline &\ \ 3\cdot 203025\\ &=\log^{-1}1596\end{aligned}$$

Thus the accumulated amount of Consols is £1596.

(2).—To show that if I be the rate of interest per-cent, then the number of years in which a sum will double itself at compound interest is given approximately by the formula **In what time a principal will reproduce itself a given number of times by accumulation at compound interest.**

$$\text{Number of years required}=\frac{69}{\text{I}}.$$

We have, if i be the rate per unit corresponding to I per-cent,

$$(1+i)^n = 2$$

$$\therefore \quad n = \frac{\log_\epsilon 2}{\log_\epsilon (1+i)}$$

$$= \frac{\log_{10} 2 \times \log_\epsilon 10}{\log_\epsilon (1+i)}$$

(See *Algebra—Theory of Logarithms.*)

$$= \frac{\log_{10} 2 \times 2{\cdot}30258509}{\log_\epsilon (1+i)}$$

$$= \frac{{\cdot}3010300 \times 2{\cdot}30258509}{\log_\epsilon (1+i)}$$

$$= \frac{{\cdot}6931472}{\log_\epsilon (1+i)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Now (*Algebra—Exponential Series*) we know that $\log_\epsilon(1+i)$ $= i - \frac{i^2}{2} + \frac{i^3}{3} - \text{\&c.}$

If we take the first term of this series as our first approximation to the value of $\log_\epsilon(1+i)$, we have

$$n = \frac{{\cdot}6931472}{i}$$

$$\doteq \frac{69{\cdot}31472}{\text{I}} \quad \text{(since I} = 100i\text{)}.$$

It is evident that the larger the value of I (or i) the less accurate is this formula, inasmuch as the neglected terms of the series for $\log_\epsilon(1+i)$ become more important the larger the value of i. An indication of the accuracy of this formula will be gathered from the following table:—

Rate of Interest per-cent.	$\frac{69}{\text{I}}$	$(1+i)^{\frac{69}{\text{I}}}$
I = 1 = $100i$	69·	1·986
„ 2 „	34·5	1·980
„ 3 „	23·	1·973
„ 4 „	17·25	1·967
„ 5 „	13·8	1·961
„ 6 „	11·5	1·955
„ 7 „	9·857	1·948
„ 8 „	8·625	1·942
„ 9 „	7·667	1·936
„ 10 „	6·900	1·930

Thus we see that when the rate of interest is 1 per-cent, a sum amounts to 1·986 times that sum in $\frac{69}{I}$ years, or nearly double, whereas at, say, 9 per-cent, it only amounts to 1·936 times the original sum in $\frac{69}{I}$ years.

Reverting to (1), we have, approximately,

$$n=\frac{\cdot 693}{\log_\epsilon(1+i)}=\frac{\cdot 693}{i\left(1-\frac{i}{2}+\frac{i^2}{3}-\&c.\right)}$$

$$=\frac{\cdot 693}{i}\left(1+\frac{i}{2}\right) \text{ approximately}$$

$$=\frac{\cdot 693}{i}+\frac{\cdot 693}{2} \qquad \text{,,}$$

$$=\frac{69\cdot 3}{I}+\cdot 35 \qquad \text{,,}$$

Thus $\frac{69\cdot 3}{I}$ with a constant addition of ·35, no matter what the value of I, will be found to give n very nearly.

Applying the method just indicated to obtain the value of n in the equation,

$$(1+i)^n=m\,;$$

that is, to find the time when a sum will become m times the original amount by accumulation at compound interest, we should have

$$n=\frac{\log_\epsilon m}{i}+\frac{\log_\epsilon m}{2}, \text{ approximately.}$$

(3).—Let the sums S_1, S_2, $S_3 \ldots S_n$ be respectively payable x_1, x_2, $x_3 \ldots x_n$ years hence: it is required to find the number of years hence when, if all the sums be then paid down, neither borrower nor lender will lose or gain by the alteration. **On the equation of payments.**

Let the required number of years be denoted by x, and let i be the rate of interest allowed in the transaction.

Then we have

$$(S_1+S_2+S_3+\ldots+S_n)v^x=\left\{\begin{array}{l}\text{Present value of all the sums}\\ \text{if paid at end of } x \text{ years.}\end{array}\right.$$

Also,

$$S_1v^{x_1}+S_2v^{x_2}+S_3v^{x_3}+\ldots+S_nv^{x_n}=\left\{\begin{array}{l}\text{Total present value of the}\\ \text{sums payable at the end}\\ \text{of } x_1,\ x_2,\ x_3 \ldots x_n \text{ years}\\ \text{respectively.}\end{array}\right.$$

By the conditions, these present values are to be equal. Hence, for the determination of x, we have the equation

$$(S_1+S_2+S_3+\ldots+S_n)v^x=S_1v^{x_1}+S_2v^{x_2}+S_3v^{x_3}+\ldots+S_nv^{x_n}.$$

Let
$$S_1+S_2+S_3+\ldots+S_n=S$$
$$S_1v^{x_1}+S_2v^{x_2}+S_3v^{x_3}+\ldots+S_nv^{x_n}=V.$$

Then we have
$$Sv^x=V$$
$$\therefore \quad x=\frac{\log V-\log S}{\log v}.$$

A rough approximation to the determination of the value of x, called the "equated time," may be obtained as follows:—

$$Sv^x=S(1+i)^{-x}=S(1-ix) \quad \text{approximately,}$$
$$S_1v^{x_1} \qquad =S_1(1-ix_1) \qquad ,,$$

and so on.

Hence we have

$$S(1-ix)=S_1(1-ix_1)+S_2(1-ix_2)+\ldots+S_n(1-ix_n);$$
$$\therefore \quad x=\frac{S_1x_1+S_2x_2+S_3x_3+\ldots+S_nx_n}{S}$$
$$=\frac{S_1x_1+S_2x_2+S_3x_3+\ldots+S_nx_n}{S_1+S_2+S_3+\ldots+S_n}.$$

That is, if each sum be multiplied by the number of years hence that it is due, the products added together and divided by the total sum due, the quotient is approximately the "equated time."

CHAPTER II.

On accumulations at different rates of interest, according as the period is yearly or otherwise.

(1).—Required to find the amount of an annuity for n years, payable quarterly in advance, supposing that the investments made at the beginning of each year are at the rate i, payable yearly, and those made in the intervals are at the rate j, payable quarterly.

The amount available for investment at the rate i at the end of the first year is evidently

$$=\frac{1}{4}\left\{\left(1+\frac{j}{4}\right)^4+\left(1+\frac{j}{4}\right)^3+\left(1+\frac{j}{4}\right)^2+\left(1+\frac{j}{4}\right)\right\}$$

$$= \text{B, say.}$$

Now, B invested for $n-1$ years at the rate i will amount to $B(1+i)^{n-1}$.

The amount available for investment at the rate i at the end of each of the succeeding years is evidently the same as for the first year, that is, B.

Hence the total accumulation at end of n years will be

$$=B\{(1+i)^{n-1}+(1+i)^{n-2}+\ldots+1\}$$

$$=B\frac{(1+i)^n-1}{i}.$$

(2).—Given the time in which a debt bearing interest is discharged by given annual instalments: show how to find in how much less time the debt will be discharged if the instalments are payable m times a year.

On the time required to discharge a debt, according as the instalments are yearly or otherwise.

Let $P_{\overline{n}|}$ denote the amount payable annually to redeem the debt in n years,

and $P_{\overline{x}|}^{(m)}$ denote the amount payable annually by m instalments to redeem the debt in x years.

Now by the terms of the question we have

$$P_{\overline{n}|} = P_{\overline{x}|}^{(m)};$$

But (Art. 26) $$P_{\overline{n}|} = \frac{1}{\frac{(1+i)^n-1}{i}}$$

$$P_{\overline{x}|}^{(m)} = \frac{1}{\frac{\left(1+\frac{i}{m}\right)^{mx}-1}{i}}.$$

Hence we have $$(1+i)^n=\left(1+\frac{i}{m}\right)^{mx};$$

$$\therefore \quad x=\frac{n\log(1+i)}{\log\left(1+\frac{i}{m}\right)^m}.$$

It will be noticed that in this example it is assumed that interest is convertible m times a year when the instalments of annuity are so paid.

On the time in which a loan will be repaid at a given annual rate of redemption.

(3).—If a loan is borrowed at a given rate of interest, and a per unit on the sum borrowed is annually applied towards paying the interest of such loan and towards discharge of the principal, to find the time in which the loan will be repaid.

Here a is the total annual payment, so that if we deduct the interest i, we have left $a-i$ as the annual payment to redeem a debt of 1.

Hence we have $a-i=P_{\overline{n}|}$

$$=\frac{1}{\frac{(1+i)^n-1}{i}};$$

$$\therefore \quad (1+i)^n-1=\frac{i}{a-i};$$

$$\therefore \quad (1+i)^n=\frac{a}{a-i};$$

$$\therefore \quad n=\frac{\log\frac{a}{a-i}}{\log(1+i)}.$$

On the accumulation of an annuity until the first payment has doubled itself.

(4).—An annuity-certain payable in advance is to accumulate until the first payment has doubled itself: to what sum will the whole annuity then amount?

If x denote the number of years when the first payment will have doubled itself,

Then we have $(1+i)^x=2.$

$$\left.\begin{array}{l}\text{The amount of the annuity pay-}\\ \quad\text{able in advance for } x \text{ years}\end{array}\right\}=\frac{(1+i)^x-1}{i}(1+i)$$

$$=\frac{1+i}{i} \text{ since } (1+i)^x=2$$

$$=\left\{\begin{array}{l}\text{present value of a perpetuity}\\ \quad\text{payable in advance.}\end{array}\right.$$

It will thus be seen that at the end of the x years there will be a sum in hand sufficient to buy a perpetuity payable in advance.

From general considerations, it will be evident that this should be so. For of the 2 representing the accumulation of the payment made at the beginning 1 may be considered as the first instalment of the perpetuity due, and the other 1 may be considered as put aside to accumulate for another x years, by the end of which time it will have amounted to 2, when again the same process of considering 1 as an instalment of the perpetuity due, the other 1 being set aside for accumulation.

It must be carefully borne in mind, in considering questions of this nature, that the epoch of reference, that is, the date to which all the values apply, is the date when the annuity has accumulated, not the present time; consequently it follows, in the example just given, that the present value of the perpetuity dates not from the present time but the end of the x years, when the annuity has accumulated according to the terms of the question.

(5).—Required to show how the amounts and present values of annuities payable more often than once a year may be deduced from the amounts and present values when payable yearly, the effective rate of interest being the same. Referring to Art. (40), we see that if an annuity be payable m times a year but interest be only convertible once a year, then

On the connection between amounts and present values of annuities payable once a year or more often, the effective rate of interest being the same.

$$\text{Amount of annuity for } n \text{ years} = \frac{1}{m}\,\frac{(1+i)^n-1}{(1+i)^{\frac{1}{m}}-1};$$

$$\text{Present value of do.} = \frac{1}{m}\,\frac{1-v^n}{(1+i)^{\frac{1}{m}}-1}.$$

Hence,

$$\text{Amount of annuity} = \frac{(1+i)^n-1}{i}\cdot\frac{\frac{i}{m}}{(1+i)^{\frac{1}{m}}-1};$$

$$\text{Present value of do.} = \frac{1-v^n}{i}\cdot\frac{\frac{i}{m}}{(1+i)^{\frac{1}{m}}-1}.$$

We notice here that the amounts and present values of annuities payable m times a year but interest only convertible once a year, are obtained from the amounts and present values of

annuities payable once a year and interest convertible once a year, by multiplication of the factor $\dfrac{\frac{i}{m}}{(1+i)^{\frac{1}{m}}-1}$. If, therefore, tables of amounts and present values of annuities payable once a year and interest convertible once a year be given, the amounts and present values of annuities payable m times a year, but interest only convertible once a year, can be readily obtained, the same factor applying to both amounts and present values of annuities and depending only upon the value of m, being independent of the number of years the annuity has to run.

A table of the logarithms of these factors for various values of i and m will be found in vol. xxiii., p. 183, of the *Journal of the Institute of Actuaries.*

Let us take the case where $m=2$, that is, the annuity payable half-yearly.

$$\text{Then the factor is } \frac{\frac{i}{2}}{(1+i)^{\frac{1}{2}}-1}=\frac{i}{2}\,\frac{(1+i)^{\frac{1}{2}}+1}{(1+i)-1}$$

$$=\frac{(1+i)^{\frac{1}{2}}+1}{2}.$$

Again, let us take the case where $m=4$, that is, the annuity payable quarterly.

$$\text{Then the factor is } \frac{\frac{i}{4}}{(1+i)^{\frac{1}{4}}-1}=\frac{i}{4}\,\frac{(1+i)^{\frac{1}{4}}+1}{(1+i)^{\frac{1}{2}}-1}$$

$$=\frac{i}{4}\,\frac{\{(1+i)^{\frac{1}{4}}+1\}}{(1+i)^{\frac{1}{2}}-1}\cdot\frac{(1+i)^{\frac{1}{2}}+1}{(1+i)^{\frac{1}{2}}+1}$$

$$=\frac{i}{4}\,\frac{\{1+(1+i)^{\frac{1}{4}}+(1+i)^{\frac{1}{2}}+(1+i)^{\frac{3}{4}}\}}{(1+i)-1}$$

$$=\frac{1+(1+i)^{\frac{1}{4}}+(1+i)^{\frac{1}{2}}+(1+i)^{\frac{3}{4}}}{4}.$$

When $m=\infty$, it will be seen that the factor is $\frac{i}{\delta}$. (Art. 40.)

If we refer to Art. (40), it will be at once seen how these constant factors arise.

Thus, Amount of annuity

$$=\frac{1}{m}\left\{1+(1+i)^{\frac{1}{m}}+(1+i)^{\frac{2}{m}}+\ldots+(1+i)+(1+i)^{1+\frac{1}{m}}+\ldots\right.$$
$$\left.+(1+i)^{n-\frac{1}{m}}\right\}$$

$$=\frac{1}{m}\left\{1+(1+i)^{\frac{1}{m}}+(1+i)^{\frac{2}{m}}+\ldots+(1+i)^{\frac{m-1}{m}}\right\}\left\{1+(1+i)+(1+i)^2\right.$$
$$\left.+\ldots+(1+i)^{n-1}\right\}$$

$$=\frac{1}{m}\left\{1+(1+i)^{\frac{1}{m}}+(1+i)^{\frac{2}{m}}+\ldots+(1+i)^{\frac{m-1}{m}}\right\}\cdot\frac{(1+i)^n-1}{i}.$$

This denotes that the annuity may be considered to be an annuity payable only once a year where the amount of each annual payment is

$$\frac{1}{m}\left\{1+(1+i)^{\frac{1}{m}}+(1+i)^{\frac{2}{m}}+\ldots+(1+i)^{\frac{m-1}{m}}\right\}.$$

On extensions or renewals of leases.

(6).—Annuity-values may be used in connection with the renewal of leases of property. Thus, suppose that an individual has secured by lease, for a term of years, possession of property bringing in a net rental of a given amount, and that after a certain number of years of his lease have run out, he wishes to extend his lease so as to prolong his possession of the property for a certain number of years after the time when the original lease expires. It is required to ascertain what sum he should pay for this extension or renewal of lease.

Let I denote the net annual amount yielded by the property.
,, n ,, the number of years of the original lease.
,, t ,, the number of years of the original lease which have already expired.
,, x ,, the number of years for which the lease is to be extended or renewed.

Then, if i be the rate of interest to be assumed, we have still, assuming I to be the net annual amount yielded by the property,

Value of annual payment of I to run for x years $=\dfrac{1-v^x}{i}\,\mathrm{I}=\mathrm{I}a_{\overline{x}|}$.

This is on the assumption that the term of x years commences at once; but as $n-t$ years have to run before the term of x years

commences, the present value is really that of an annuity for x years to commence at the expiration of $n-t$ years.

That is,

$$\left.\begin{array}{l}\text{Value of annual payment of I to run for } x\\ \quad\text{years after the termination of } n-t \text{ years}\end{array}\right\} = v^{n-t}\mathrm{I}a_{\overline{x}|} \quad \text{(Art. 31)}$$

$$= \frac{v^{n-t}-v^{n+x-t}}{i}\mathrm{I}$$

$$= (a_{\overline{n+x-t}|}-a_{\overline{n-t}|})\,\mathrm{I}.$$

Again, we may take the case where a property is held on lease for a term of n years, the lessee having the right at the end of n years to renew his lease for another term of n years, subject only to the payment of a sum down called a fine. These fines are generally imposed in place of increasing the rent, where the property is considered to be of greater annual value than the rent imposed, and it will of course follow that they may vary in amount from one term of lease to the next.

If we suppose the fine imposed at the end of the first, second, third . . . mth term of years to be F_1, F_2, F_3 . . . F_m, and desire to estimate the present value of the next m fines when there is still unexpired $n-t$ years of the first lease, we see that such present value is given by the formula

$$\text{Present value of next } m \text{ fines} = v^{n-t}\{F_1+F_2v^n+F_3v^{2n}+\ldots+F_m{}^{(m-1)n}\}.$$

If F_1, F_2 . . F_m be assumed to be all of the same amount, say F, we have

$$\text{Present value of next } m \text{ fines} = Fv^{n-t}\{1+v^n+v^{2n}+\ldots+v^{(m-1)n}\}$$

$$= Fv^{n-t}\frac{1-v^{mn}}{1-v^n}.$$

If, further, we assume m to be infinite, we have

$$\text{Present value of the fines in perpetuity} = \frac{Fv^{n-t}}{1-v^n}.$$

On annuities with different intervals for payment and conversion.

(7).—In Art. (38) it has been shown that the value of a perpetuity is the same no matter how often interest is convertible, provided the payments of the perpetuity are made simultaneously with the conversions of interest.

In other words, if x be the nominal rate of interest, then $a_{\infty}^{(m)} = \frac{1}{x}$.

As, therefore, a perpetuity is of the same value however often payment and interest may be subdivided, it follows that two annuities, with different intervals for payment and conversion, must show a turning-point, before which the present value of all the payments made in one year in the case of one annuity exceeds the present value of all the payments, made in the same year, in the case of the other annuity, and after which the present value of the payments in one year, in the case of the first annuity, falls short of the present value of the payments in the same year in the case of the second annuity.

Take the cases where interest is convertible and annuity payable once a year and m times a year respectively.

In the tth year the value of the payments made are respectively v^t and $\dfrac{\left(1+\frac{i}{m}\right)^m-1}{i} \cdot \dfrac{1}{\left(1+\frac{i}{m}\right)^{mt}}$. If these are equal we have

$$\frac{1}{(1+i)^t} = \frac{\left(1+\frac{i}{m}\right)^m-1}{i} \cdot \frac{1}{\left(1+\frac{i}{m}\right)^{mt}}$$

whence
$$t = \frac{\log\left\{\left(1+\frac{i}{m}\right)^m-1\right\}-\log i}{\log\left(1+\frac{i}{m}\right)^m-\log(1+i)}.$$

Now, De Morgan has shown, in the *Journal of the Institute of Actuaries*, vol. xii., p. 207, that the above result may be put into the form

$$t=\frac{1}{i}+\frac{3m+1}{4m}+\frac{5(m-1)}{12m^2}i+\ldots$$

and this series is always less than $\frac{1}{i}+1$ when i is less than unity.

The example given by De Morgan is as follows:—$i=\cdot 10$, so that $\frac{1}{i}+1=11$.

Year.	Present Value of Payments made in Year		
	$m=5$.	$m=2$.	$m=1$.
1	·9427	·9297	·9091
9	·4269	·4259	·4241
10	·3867	·3863	·3855
11	·3502	·3504	·3505
12	·3172	·3178	·3186

Here we see that as far as 10 years the value of the payments is greater the greater the value of m, but in the 11th and following years, the greater the value of m the less is the value of the payments.

It is to be particularly remarked that the formula for determining the turning-point is independent of m.

Another curious property to be deduced from this result is that for deferred annuities, where the term t for which they are deferred is not less than the turning-point just determined by the formula

$$t=\frac{1}{i}+\frac{3m+1}{4m}+\frac{5(m-1)}{12m^2}i+\ldots,$$

the greater the value of m the less will be the value of the deferred annuity.

Where, however, the term for which the annuities are deferred is less than t, we cannot say beforehand whether the value of the deferred annuity will be less or greater according as m varies, except in the case where the term for which the annuity is deferred and the term for which it is to vary are together less than t. In this case it is clear that since the value of the payments made in any one year is always greater the greater the value of m, the deferred annuity must necessarily be greater the greater the value of m.

As regards immediate annuities it is evident, without any demonstration, that the greater is m the greater also is the value of the annuity, since an immediate annuity for n years + a perpetuity deferred n years always amount together to the same value, namely, an immediate perpetuity whose value is independent of m.

8.—Suppose it is required to find the value of a series of annual payments in geometric progression. For instance, let the first payment, due at the end of a year, be A; the second, due at the end of 2 years, A^2; and so on: that due at the end of n years being A^n. Then, if the rate of interest to be used in determining their present value is i, we have

On the value of a series of payments in geometric progression.

$$\left.\begin{array}{l}\text{Present value of the}\\ \quad n \text{ payments}\end{array}\right\} = Av + A^2v^2 + A^3v^3 + \ldots + A^nv^n \quad . \quad . \quad (1)$$

$$= X + X^2 + X^3 + \ldots + X^n \quad . \quad . \quad . \quad . \quad (2)$$

where $X = Av$.

If, now, $X = Av = \frac{A}{1+i}$ be taken to be equal to $\frac{1}{1+I}$, we see that the series (1) is the value of an annuity-certain for n years, where the annual payments are each 1 in amount, at a rate of interest I, such that $\frac{A}{1+i} = \frac{1}{1+I}$.

Hence it follows that the value of I is $\frac{1+i}{A} - 1$, and consequently the value of the series of n payments is that of an ordinary annuity-certain for n years, calculated at the rate of interest $\frac{1+i}{A} - 1$.

Let us suppose $A = 1 + j$, so that any payment, say the tth, is $(1+j)$ times the $(t-1)$th; in other words, that

$$A^t = (1+j)A^{t-1}$$
$$= A^{t-1} + jA^{t-1}.$$

Further, let the series of annual payments be in perpetuity. Then their value is known to be $\frac{1}{I}$ where $I = \frac{1+i}{A} - 1$

$$= \frac{1+i}{1+j} - 1$$

$$= \frac{i-j}{1+j}.$$

Hence the value of such a series of payments in perpetuity is $\frac{1+j}{i-j}$. Now $\frac{1}{i-j}$ is the value of a perpetuity of 1 at the rate of

interest $i-j$, and $1+j=A$ is the first payment of the increasing perpetuity. Hence it follows that the value (calculated at a rate of interest i) of an increasing perpetuity where the rate of increase in the payments is j, is equal to the value of a perpetuity where the payments are all of the same amount, namely, $1+j$, that of the first payment of the increasing perpetuity, calculated at a rate of interest equal to the difference between the original rate of interest i, and the rate of increase in the payments j.

CHAPTERS III. AND IV.

Example illustrating the direct application of Makeham's formula.

(1).—A bond securing an advance of 1 payable at the end of n years, with interest in the meantime at the rate i, is to be repaid by equal instalments of $\frac{1}{n}$. Required the price to be paid for the bond to yield the purchaser interest at the rate i'.

Here $\quad C'$ (Art. 67) = present value of capital receivable

$$=\frac{1}{n}\left\{v'+v'^2+v'^3+\ldots v'^n\right\}$$

$$=\frac{a_{\overline{n}|}}{n};$$

$$\therefore\quad 1-A=\frac{i'-i}{i'}(1-C')$$

$$=\frac{i'-i}{i'}\left(1-\frac{a_{\overline{n}|}}{n}\right);$$

$$\therefore\quad \left.\begin{array}{l}\text{A, or price to be paid}\\ \quad\text{for the bond}\end{array}\right\}=1-\frac{i'-i}{i'}\left(1-\frac{a_{\overline{n}|}}{n}\right).$$

Example illustrating the application of Gray's formula and Makeham's formula to the determination of the rate of interest in the case of a loan where the sum repaid differs from that advanced.

(2).—In this case it is proposed to give at length a practical illustration both of Mr. Makeham's formula in Chapter IV., and of Mr. Peter Gray's formula in Chapter III.

Mr. Gray thus states and deals with the problem in question (*see* vol. xiv. of the *Journal of the Institute of Actuaries*, p. 182) :—

In the Austrian Loan of 1865 the conditions are as follows:—

734,694 bonds, each of £19. 17*s*., issued at £13. 14*s*. 4*d*., that is, at a discount of £6. 2*s*. 8*d*. each.

£1	0	0	per bond to be paid on application, say, Dec. 1, 1865, and the remaining £12. 14*s*. 4*d*. in the following instalments:—
1	19	7	on Dec. 15, 1865.
3	11	7	,, Feb. 10, 1866.
3	11	7	,, April 10, ,,
3	11	7	,, June 10, ,,
£13	14	4	Total.

Subscribers will be at liberty to pay their scrip in full on any one of the above dates, under discount at 6 per-cent per annum.

The bonds are to bear interest at the rate of 9*s*. 11*d*. each (=·4958333) per half-year (a trifle under $2\frac{1}{2}$ per-cent on the *nominal* amount), payment of which was to become due on the 1st June and the 1st December of each year.

The bonds are to be redeemed in thirty-seven years, 9928 (to be selected by lot) half-yearly, at the same dates as the payments of interest. The first drawing to take place in May, 1868, and the bonds then drawn to be paid off on the 1st of June thereafter.

Required the cost per-cent of the loan to the borrower, and the rates realized on the bonds paid off each half-year.

Here it will be observed that the number of the bonds being 734,694 while $9928 \times 74 = 734{,}672$, the payment of 22 bonds is left unprovided for. It is here assumed that these 22 bonds are to be included in the last payment, making the number paid off at the end of the 78th half-year 9950.

The principal points in which this problem differs from the former are that here, first, the loan is issued at a discount; secondly, the repayments of principal do not commence immediately; and thirdly, that these repayments (with the exception of the last, *see above*,) are uniform.

To determine the cost to the borrower we have to find the rate at which the values at any epoch of the sums receivable and payable by him are equal to each other. The most convenient epoch of reference, in this case as in the last, is the commencement of the transaction—the date of issue, December 1st, 1865.

The receipts are reduced to this epoch as follows:—

Dec. 1 . .	734,694·000		
„ 15 . .	1,454,081·875	14 days . .	3,346·380
Feb. 10 . .	2,629,592·275	71 „ . .	30,690·584
April 10 . .	2,629,592·275	130 „ . .	56,194·027
June 10 . .	2,629,592·275	191 „ . .	82,561·993
	10,077,552·700		172,792·984
	172,792·984		
	9,904,759·716		

The reduction is effected by discounting at 6 per-cent (the stipulated rate) the several instalments for the time to elapse between the epoch of reference and their respective dates of payment. The discount amounts to 172,793, deducting which from (£13. 14*s*. 4*d*. × 734,694 =) 10,077,553, the difference, 9,904,760, is the value at the chosen epoch of the borrower's receipts.

Next, to find the value of his payments at the same epoch. It will be convenient to consider these in three portions. The first is a uniform annuity, for four half-yearly terms, of the interest on the bonds; the second is a variable annuity deferred four terms, and to last for seventy-four, whose payments are for each term, the sum of the interest on the principal unpaid at the beginning of that term and the amount of the bonds repayable at the end of it; and the third is the amount of the 22 residual bonds repayable at the end of the seventy-eighth term. We will deal with these in order.

First, the uniform annuity. This consists of four half-yearly payments of ·495833 × 734,694 = 364,285·775; and if we denote the required half-yearly rate by i, its value will be

$$364{,}285{\cdot}775\,\frac{1-v^4}{i}\,.$$

Secondly, the variable annuity, to be entered upon in two years and to make seventy-four half-yearly payments. We shall here use the formula (D) of Art. (47), which adapted to this case is

$$v^4\left\{\frac{b_1}{i}+\frac{\Delta b_1}{i^2}+\ldots\ldots-v^{74}\left(\frac{b_{75}}{i}+\frac{\Delta b_{75}}{i^2}+\ldots\right)\right\}.$$

We shall determine b_1, &c., as follows :—

Int. on	734,694 bonds	@ ·495833	.	364,285·775
Payable	9,928 ,,	@ 19·85	.	197,070·800
				561,356·575 $= b_1$
Int. on	724,766 ,,	@ ·495833	.	359,363·142
Payable	9,928 ,,	19·85	.	197,070·800
				556,433·942 $= b_2$

It is unnecessary to go further. We see that $b_2 - b_1$ or $\Delta b_1 = -4922 \cdot 633$, which is the half-yearly interest on the 9928 bonds paid off the preceding year; and each succeeding payment will evidently have the same difference.

Hence $\quad \Delta^2 b_1$, &c. $=0$.

Hence also

$$\begin{aligned} b_{75} &= b_1 + 74\Delta b_1 \\ &= 561{,}356 \cdot 575 - 4922 \cdot 633 \times 74 \\ &= 561{,}356 \cdot 575 - 364{,}274 \cdot 867 \\ &= 197{,}081 \cdot 708 \end{aligned}$$

and $\quad \Delta b_{75} = -4922 \cdot 633.$

Hence the value of this annuity is

$$v^4 \left\{ \frac{561{,}356 \cdot 6}{i} - \frac{4922 \cdot 633}{i^2} - v^{74} \left(\frac{197{,}081 \cdot 7}{i} - \frac{4922 \cdot 633}{i^2} \right) \right\}.$$

Thirdly, the value of the 22 bonds due at the end of the 78th half-year, is

$$22 \times 19 \cdot 85 \times v^{78} = 436 \cdot 70\, v^{78}.$$

The sum of the three results, namely,

$$\frac{364{,}285 \cdot 8}{i} + \frac{197{,}070 \cdot 8 v^4}{i} - \frac{4922 \cdot 633 v^4}{i^2} - \frac{197{,}081 \cdot 7 v^{78}}{i} + \frac{4922 \cdot 633 v^{78}}{i^2} + 436 \cdot 7 v^{78},$$

or

$$\frac{364{,}285 \cdot 8}{i} + \frac{197{,}070 \cdot 8}{i(1+i)^4} - \frac{4922 \cdot 633}{i^2(1+i)^4} - \frac{197{,}081 \cdot 7}{i(1+i)^{78}} + \frac{4922 \cdot 633}{i^2(1+i)^{78}} + \frac{436 \cdot 7}{(1+i)^{78}},$$

is the value of the borrower's payments at the epoch of reference, in terms of i. And we have now to find the value of i which makes this expression equal to 9,904,760. This is done by trial, and the operation is as follows :—

i	·04	·045	·0437	·0436473	·0436484
$\log i$	$\bar{2}$·6020600	$\bar{2}$·6532125	$\bar{2}$·6404814	$\bar{2}$·6399574	$\bar{2}$·6399683
„ $(1+i)$	0·0170333	0·0191163	0·0185757	0·0185538	0·0185542
„ $(1+i)^4$	0·0681334	0·0764652	0·0743027	0·0742150	0·0742168
„ $i\,(1+i)^4$	$\bar{2}$·6701934	$\bar{2}$·7296777	$\bar{2}$·7147841	$\bar{2}$·7141724	$\bar{2}$·7141851
„ $i^2(1+i)^4$	$\bar{3}$·2722534	$\bar{3}$·3828902	$\bar{3}$·3552655	$\bar{3}$·3541298	$\bar{3}$·3541534
„ $(1+i)^{78}$	1·3286005	1·4910706	1·4489030	1·4471933	1·4472285
„ $i(1+i)^{78}$	$\bar{1}$·9306605	0·1442831	0·0893844	0·0871507	0·0871968
„ $i^2(1+i)^{78}$	$\bar{2}$·5327205	$\bar{2}$·7974956	$\bar{2}$·7298658	$\bar{2}$·7271081	$\bar{2}$·7271651
364285·8	5·5614423	5·5614423	5·5614423	5·5614423	5·5614423
	$\bar{2}$·6020600	$\bar{2}$·6532125	$\bar{2}$·6404814	$\bar{2}$·6399574	$\bar{2}$·6399683
A	6·9593823	6·9082298	6·9209609	6·9214849	6·9214740
197070·8	5·2946223	5·2946223	5·2946223	5·2946223	5·2946223
	$\bar{2}$·6701934	$\bar{2}$·7296777	$\bar{2}$·7147841	$\bar{2}$·7141724	$\bar{2}$·7141851
B	6·6244289	6·5649446	6·5798382	6·5804499	6·5804372
4922·633	3·6921974	3·6921974	3·6921974	3·6921974	3·6921974
	$\bar{3}$·2722534	$\bar{3}$·3828902	$\bar{3}$·3552655	$\bar{3}$·3541298	$\bar{3}$·3541534
C	6·4198440	6·3093072	6·3369319	6·3380676	6·3380440
197081·7	5·2946463	5·2946463	5·2946463	5·2946463	5·2946463
	$\bar{1}$·9306605	0·1442831	0·0893844	0·0871507	0·0871968
D	5·3639858	5·1503632	5·2052619	5·2074956	5·2074495
4922·633	3·6921974	3·6921974	3·6921974	3·6921974	3·6921974
	$\bar{2}$·5327205	$\bar{2}$·7974956	$\bar{2}$·7298658	$\bar{2}$·7271081	$\bar{2}$·7271651
E	5·1594769	4·8947018	4·9623316	4·9650893	4·9650323
436·7	2·6401832	2·6401832	2·6401832	2·6401832	2·6401832
	1·3286005	1·4910706	1·4489030	1·4471933	1·4472285
F	1·3115827	1·1491126	1·1912802	1·1929899	1·1929547

i	·04	·045	·0437	·0436473	·0436484
A	9107100	8095242	8336062	8346126	8345916
B	4211400	3672353	3800477	3805833	3805723
C	2629900	2038484	2172360	2178040	2177930
D	231200	141372	160421	161248	161231
E	144400	78470	91692	92276	92264
F	20	14	16	16	16
	13462920	11846079	12228247	12244251	12243919
	2861100	2179856	2332781	2339288	2339161
Result	10601820	9666223	9895466	9904963	9904758
	9904760	9904760	9904760	9904760	9904760
Error	697060	238537	9294	203	2

We commence our trials with $i = {}$·04, conjecturably near the truth. It turns out to be too small, giving an error of 697060. We then try ·045, which is too great—error, 238537. We then use *False Position*, which, as applied to problems such as the present, gives the following rule:—Multiply the error of the last result by the last correction of the rate, and divide the product by the difference of the last two results. The quotient will be a new correction to be applied in either augmentation or diminution of the rate last used, according as that rate was found to be too small or too great.

We thus have here

$$\frac{238537 \times {}·005}{10601820 - 9666223} = \frac{1192·685}{935597} = {}·0013^* \text{ (or, more accurately, ·00127.)}$$

And ·045 − ·0013 = ·0437 is a corrected rate (more accurately, ·04373). This is found on trial to be still too great, but the error is reduced to 9294. For a new correction, we have

$$\frac{9·294 \times {}·0013}{9895 - 9666} = \frac{·01208}{229} = {}·0000527.$$

* As the quotients in this operation are taken to only two or three places, it is unnecessary to use more than three or four figures in the numbers producing them. Thus, instead of the above, we might have written

$$\frac{238·5 \times {}·005}{10602 - 9666} = \frac{1·193}{936} = {}·0013.$$

Hence $\cdot 0437 - \cdot 0000527 = \cdot 0436473$ is a second corrected rate, trial of which shows that it is a trifle too small, the error being 203.

It would hardly be considered necessary to seek a closer approximation than this. We have nevertheless, as a matter of curiosity, carried the operation a step further. Thus:—

$$\frac{203 \times \cdot 0000527}{9904963 - 9895466} = \frac{\cdot 01070}{9497} = \cdot 0000011.$$

This gives $\cdot 0436473 + \cdot 0000011 = \cdot 0436484$ for a third corrected rate; and trial of this gives, as shown, an error of 2.* It thus appears that the method employed enables us by three or four trials to assign the required rate to any degree of approximation that may be desired.

It is the half-yearly rate that has just been determined. The yearly rate is easily deduced from it, thus:—

$$(1 \cdot 0436484)^2 - 1 = \cdot 0892018, \text{ the yearly rate;}$$

that is, 8·92018 per-cent.

In vol. xviii., p. 137, of the *Journal of the Institute of Actuaries*, Mr. Makeham thus deals with the same problem.

In the Austrian Loan of 1865, the capital to be received by the purchasers of the bonds amounted to £14,583,675·9,—payable by 74 equal half-yearly instalments of £197,070·8 each, commencing at the expiration of $2\frac{1}{2}$ years,—with an additional payment of £436·7 at the termination of the period to make up the required amount, interest at the rate of 2·4979 per-cent half-yearly being payable during the currency of the loan. The consideration paid by the lenders was £9,904,760.

The value of the capital receivable is

$$a_{\overline{74}|} \times v^4 \times 197{,}070 \cdot 8 + v^{78} \times 436 \cdot 7.$$

Taking as the first trial-rate 4 per-cent, and dividing by 100 to save unnecessary figures, the calculation of the expression $i' \cdot \frac{C-A}{C-C'}$ (Arts. 67, 68) is as follows:—

* In making the trial, it is in this case necessary to form $\log(1+i)$ to 9 places in order to have $\log(1+i)^{78}$ true in the seventh place.

$\log a_{\overline{74}|} = 1{\cdot}37343$ $\quad C = 145{,}837$ $\quad \therefore C - A = 46{,}789$ $\quad \log (C-A) = 4{\cdot}67014$

$\log v^4 = \bar{1}{\cdot}93187$ $\quad A = 99{,}048$ $\quad C - C' = 106{,}034$ $\quad \log (C-C') = 5{\cdot}02543$

$1970{\cdot}708 = 3{\cdot}29462$ $\quad C' = 39{,}803$ $\quad \bar{1}{\cdot}64471$

$\log C' = 4{\cdot}59992$

$$\therefore \frac{C-A}{C-C'} = {\cdot}44128$$

$$\therefore i' \cdot \frac{C-A}{C-C'} = {\cdot}04 \times {\cdot}44128$$

$$= {\cdot}01765$$

$$i = {\cdot}02498$$

$$\therefore i + i'\frac{C-A}{C-C'} = i' = {\cdot}04263$$

…TE.—As the value of the final payment, $v^{78} \times 4{\cdot}367$, has evidently no significant effect upon the value of C′, it has been neglected.

From this it appears that the trial rate taken is too small, and that the true rate is something over 4¼ per-cent. Repeating the calculation, substituting 4¼ per-cent instead of 4 per-cent, it will be found that we get $i' = {\cdot}04333$.

Having found two approximations, we may proceed to find a third as follows:—

	4·	gives	4·263
	4·25	„	4·333
Differences	·25		·07

Hence $4 + x$ gives $4{\cdot}263 + \frac{{\cdot}07}{{\cdot}25}x$ nearly; whence we find, equating these two values and solving the equation in x,

$$x = \frac{{\cdot}263 \times {\cdot}25}{{\cdot}25 - {\cdot}07} = {\cdot}365.$$

whence $4 + x = 4{\cdot}365$, which is true to the last figure (see Mr. Gray's solution given above).

Mr. Makeham claims for this method that it gives, in two trials, as accurate a result as that obtained by Mr. Gray's method by a greater number of trials and much more extensive calculation.

On other methods of approximating to the rate of interest on loans where the amount repaid differs from the amount advanced.

(3).—Besides the methods mentioned in Chapter IV., of determining the actual rate of interest paid in the case of a loan issued at one price and repaid at another, several others have from time to time been suggested. One method is, to ascertain the "probable epoch"—that is, the time which will elapse before just half the

loan is repaid, and to consider the whole loan as repaid at that date, and then ascertain the actual rate of interest which would be paid on this assumption. This method is clearly incorrect in theory, and as practically used gives results wide of the mark.

In symbols, it is as follows:—

If s be the sinking fund per 1, set aside the first year, and n the duration of the loan,

then $$s\frac{\overline{1+i}|^{n}-1}{i}=1\,;$$

and if $$s\frac{(1+i)^{t}-1}{i}=\tfrac{1}{2},$$

t is the "*probable epoch*".

To find t we have $$s\frac{\overline{1+i}|^{t}-1}{i}=\tfrac{1}{2},$$

but $$s=\frac{i}{\overline{1+i}|^{n}-1}\,;$$

$$\therefore\quad \frac{\overline{1+i}|^{t}-1}{\overline{1+i}|^{n}-1}=\tfrac{1}{2},$$

or $$\overline{1+i}|^{t}=\frac{\overline{1+i}|^{n}+1}{2},$$

whence t can be found.

Denoting by x the actual rate of interest paid, we have, if loan is redeemed at par and issued at discount of D, say

$$D=\frac{1-(1+x)^{-t}}{x}\,(x-i)\,.$$

Another method, suggested as an improvement upon the preceding, is as follows:—The loan being repayable in n yearly instalments, increasing year by year in the ratio 1 to $1+i$ (i being the nominal rate of interest at which the loan is issued), then denoting the amount paid off the first year by s, the following amounts will be

$$s(1+i),\ s(1+i)^{2},\ \ldots\ldots\ s(1+i)^{n-1}\,;$$

and the average amount paid off in a year

$$=\frac{s+s(1+i)+\ .\ .\ .\ .\ +s(1+i)^{n-1}}{s}=\frac{1}{n},$$

and putting $s\ (1+i)^{t-1}=\frac{1}{n}$, then at the end of the tth year $\frac{1}{n}$th of the whole loan will be repaid. This year is called the "epoch of mean probability." Then an approximation is made to the actual rate of interest paid on the loan by considering the whole loan as repaid t years from date of issue.

This method, like the preceding, is manifestly incorrect, but it gives values nearer to the true values than the former. The first of the two just-mentioned methods of approximation is in actual use; and the second, which has been suggested as an improvement on the first, appears to be due to Messieurs Vintégoux and de Reinach, *Formules et Tables d'Intérêts composés et d'Annuités*, (Paris, 1874), pp. 34–36.

On the rate of interest made by investors in loans when the amount repaid differs from that advanced.

(4).—Besides the question of the rate of interest paid by the borrower, there is the no less important question of the rate of interest realized by the investor. Of course, if the whole loan were taken up by a single individual, the rate of interest would in each case be the same. In practice it may be said this is never the case, and, as far as can be gathered, the general impression seems to be that, in the case, say, of the purchaser of a single bond, it is impossible to say what rate of interest he will realize on his investment, as his bond may be paid off in any year of the n years which repayment extends, the rate of interest he realizes becomes, in the general case of repayment, at par, the bond being bought at a discount, less and less as repayment is deferred; and on the contrary, where the bonds are issued or bought at a premium, the rate of interest realized by the investor is greater the longer repayment is deferred.

Let us consider the case of the purchaser of a single bond from another point of view. If 1 be the amount of the bond, L the amount of the loan, P the portion of the loan repaid by the accumulative sinking fund the first year, i being the nominal rate of interest on the loan, then $\frac{\mathrm{P}}{\mathrm{L}}=$ chance of the particular bond in question being drawn for repayment the first year,

$$(1+i)\,\frac{\text{P}}{\text{L}}=\text{chance for 2nd year}=(1+i)s\ \text{(say)}$$

$$(1+i)^2\frac{\text{P}}{\text{L}}=\quad ,,\quad \text{3rd}\ ,,\ =(1+i)^2s,$$

and so on.

Now at the end of the first year the investor receives his dividend i, and has a chance s of receiving 1 in repayment of his bond, and the value of this chance being $s\times 1$ or s, he may be considered as receiving $i+s$ at the end of the first year. Similarly at the end of the second year he will receive a dividend i if his bond was not drawn for repayment in the previous year, and the chance of this being $1-s$, the value of such dividend may be taken as $(1-s)i$, and the chance of the bond being drawn for repayment being $s\ (1+i)$, the total value of his expectation at end of second year $=(1-s)i+s(1+i)=i+s$. Similarly the value of his expectation at the end of the third year

$$\begin{aligned}
&=\{1-s-s(1+i)\}i+s(1+i)^2\\
&=i(1-s)-s(1+i)(i-1-i)\\
&=i(1-s)+s(1+i)\\
&=i+s,
\end{aligned}$$

and so on for other years.

In other words, the purchaser of a single bond is in exactly the same position as if he bought an annuity-certain of $i+s$ to run for the term of the loan. Assuming, therefore, that the price of issue or purchase was $1-p$ per 1, we have

$$1-p=(i+s)\,\frac{1-(1+x)^{-n}}{x},$$

x being the rate of interest at which the various payments are discounted, that is, the rate of interest realized by the purchaser; and we see that the same formula obtains as in the case of the borrower.

Additional numerical illustrations. (5).—In the table which follows, a number of the principal foreign loans brought out in England in recent years have been taken, and there has been calculated the actual rate of interest paid by the borrower in each case, first without allowing for the redemption

of the loan at par, and secondly allowing for redemption at par, the formulas used being taken from Chapter IV.

Price of Issue per-cent.	Nominal Rate of Interest per-cent.	Sinking Fund per-cent per Annum.	Actual Rate of Interest per-cent paid by Borrower, assuming			
			(1) Repayment at Issue Price.	(2) Repayment at Par.		
				By Formula (A).	By Formula (C).	By Formula (D).
72½	6	2½ accum.	8·2759	10·2005	10·1966	10·1987
88½	6	2½ „	6·7797	7·5021	7·5021	7·5021
76	6	1· „	7·8947	3·6070	8·6056	8·6055
78	6	2· „	7·6923	8·9439	8·9436	8·9438
92	7	2· „	7·6087	7·9734	7·9733	7·9734
84	6	2· „	7·1429	8·0329	8·0328	8·0329
94	5	2· „	5·3191	5·7420	5·7419	5·7420
93	7	3·87 „	7·5269	8·0071	8·0071	8·0071
90	9	3·41 „	10·0000	10·8875	10·8151	10·8464
77½	5	·083 „	6·4516	6·5263	6·5262	6·5263
78	5	·13 „	6·4103	6·5193	6·5192	6·5193

With regard to the above table, various points have to be attended to. In the first place, it will be noticed that the sinking fund is generally quoted as percentage in round numbers, so that the time required for redemption of the loan will not generally be an exact number of years. This will not in practice have any considerable effect upon the results, which are obtained by taking the term of redemption to be the nearest number of complete years.

Another point is, that in the majority of cases the interest and drawings are payable half-yearly, and in some few cases quarterly; but this, too, will be found to be not of material consequence.

Still another point to consider is, that as the bonds are usually for fixed amounts of £20, £50, £100, £500, or £1000, the actual amount of any drawing will not in general be the exact theoretical amount.

Reference may be made to vol. xix., p. 77, of the *Journal of the Institute of Actuaries.*

Further consideration of Art. (64) with special reference to the assumptions made as to the rate of interest at which accumulating sinking fund can be invested.

(6).—If we refer to formula (E_1) of Art. (64), namely,

$$p=\{i'(1-p)-i\}\,\frac{(1+i')^n-1}{i'},$$

where $1-p$ is the capital advanced, 1 the capital to be repaid at end of n years, i the interest actually received, and i' the rate of interest actually made by the lender, we see that the assumption is made that the difference between $i'(1-p)$ and i is accumulated at the same rate of interest i' as that made on the capital advanced by the lender. Cases may occur, however, where it is desired to make some other assumption as to the rate of interest at which the difference between $i'(1-p)$ and i shall be accumulated. Thus, suppose this rate to be x, so that formula (E_1) becomes

$$p=\{i'(1-p)-i\}\,\frac{(1+x)^n-1}{x},$$

Let us denote $\dfrac{(1+x)^n-1}{x}$ by $\dfrac{1}{P_{\overline{n}|}}$; then we get

$$p=\{i'(1-p)-i\}\,\frac{1}{P_{\overline{n}|}}$$

$$\therefore\quad p=(i'-i)\,\frac{1}{P_{\overline{n}|}+i'}\quad .\ .\ .\ .\ .\ (1)$$

Comparing this formula for p with formula (E) of Art. (64), we see that instead of $a'_{\overline{n}|}=\dfrac{1-v'^n}{i'}=\dfrac{1-(1+i')^{-n}}{i'}$ we have $\dfrac{1}{P_{\overline{n}|}+i'}$, which denotes the value of an annuity for n years such that the investor is to make interest at the rate i' on his entire investment throughout the n years, and the accumulating sinking fund $P_{\overline{n}|}$ to reproduce the capital at the end of the n years is to bear interest at another rate of interest x. (See Art. 27.)

We may obtain from (1) another formula. Thus, from (1),

$$p(P_{\overline{n}|}+i')=i'-i$$

$$\therefore\quad i'(1-p)=i+pP_{\overline{n}|}.$$

Here $pP_{\overline{n}|}$ is the difference between $i'(1-p)$ and i, and denotes the annual sum to be set aside to accumulate at the rate of interest x so as to amount to p at the end of the n years.

(7).—*(This Example should properly have been given as an illustration of Art. 27 of Chapter II.)* As an illustration of the practical application of the formula

On the application of the formula of Art. (27) to the valuation of mining property.

$$a_{\overline{n}|} = \frac{1}{P_{\overline{n}|} + i'} \quad \text{(see Art. 27)},$$

where $P_{\overline{n}|}$ the accumulated sinking fund $= \frac{i}{(1+i)^n - 1}$ is invested at the rate i, and the stipulation is made that the entire capital originally invested, that is, $a_{\overline{n}|}$, is to bear interest at the rate i' for the n years, the following extract from an Introductory Note by Mr. Peter Gray to the *Mining Engineers' Valuing Assistant*, by H. D. Hoskold (Longmans, 1877), will be found to be of interest:—

"The course of proceeding in the valuation of a mine appears to be as follows:—The valuator, in the exercise of his professional skill and knowledge, names a sum and a term of years, the former to be considered as the annual income to be derived from the mine, and the latter as the number of years that this income is to last. It is further arranged between the parties that the purchaser is to be allowed a specified rate of interest on his outlay during the entire term. The required value is thus presented in the form of an annuity-certain, the elements of which—the sum, the term, and the rate—are known; and there remains only the conversion of that value into a present sum.

"One of the points on which I am requested to give my opinion is as to the correct method of valuing the annuity which forms the subject of the valuator's first determination.

"Ordinarily, the valuation of an annuity for a term of years, when the rate of interest to be allowed to the purchaser has been arranged, is a sufficiently simple matter. The well-known tables of Smart (reproduced by Jones in his *Treatise on Annuities*), and others, furnish, in the cases that usually arise, all the aid that can be required, even by the most inexperienced computer. But the cases with which we have here to do are somewhat complicated by the entrance of a consideration that does not present itself—in so pressing a way, at least—in general practice.

"It cannot be doubted that the purchaser of an annuity for a term, on which he is to be allowed interest at a specified rate, ought, as regards this transaction, to be in the same position, pecuniarily, at the end of the term, as if he had lent his money during the term at the same rate. The lender receives his interest annually, and has the sum lent returned at the end of the term.

"But the purchaser of an annuity must recoup himself by investing the excess of his annuity over the annual interest on his outlay, at such a rate that at the end of the term his capital will be reproduced.

"The lowest rate at which this reproduction can be *assumed* by the vendor or purchaser to be effected, is the rate allowed in the purchase of the annuity, as will presently be shown. In the case of annuities purchased at current rates, but little inconvenience and loss will occur to the purchaser from this restriction as to the rate of re-investment, since *practicable* rates in respect of such will usually differ but little from the *stipulated* rates. In the cases with which we are here concerned, however, the state of matters is far otherwise.

"In the purchase of mining property, the purchaser, for reasons with which we have nothing here to do—they are fully discussed in the following work—is usually, perhaps always, allowed a rate of interest on his outlay far exceeding that at which he can invest the surplus of his annuity, which is called with propriety the *Redemption Fund;* and hence, if the *ordinary tables* are used in the valuation of the annuity determined and assigned by the valuator, the result must be a loss to the purchaser, more or less heavy according to circumstances, since in them the difference between the two rates is ignored. In the present connexion, therefore, special methods must be employed."

It should be added, that it would appear to be not an unusual matter for purchasers of mines to pay as purchase-money a sum calculated as above on the assumption that the sinking fund for reproduction of capital bears interest at the low rate of $2\frac{1}{2}$, 3, $3\frac{1}{2}$, or 4 per-cent, whereas the capital itself is to bear interest for the term of years agreed upon, at such high rates as 15, 20, 25 per-cent, or even higher.

CHAPTER VI.

On the Application of the Higher Mathematics to the Theory of Compound Interest.

It is not proposed to enter at any length into the subject of the application of the higher mathematics to questions involving the theory of compound interest, but simply to give some general illustrations which, to those familiar with the differential and integral calculus, will be found sufficient; and it is to be understood that for such only is this chapter intended.

Application to the determination of the amounts and values of annuities, with geometrical illustrations.

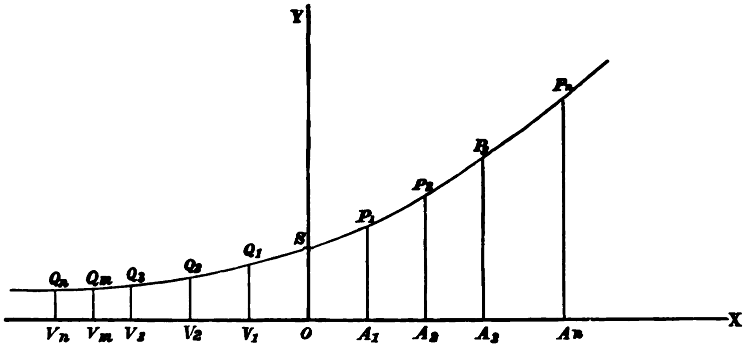
(73).—Referring to Art. (16), (Chapter I.), it is there shown that the ordinates (y) of the logarithmic curve

$$y=(1+i)^x$$

denote the amounts and present values respectively of a sum 1 due x years hence, according as x is measured to the right or left-hand side of the initial ordinate OX, taken as the axis of y, or we say that

$$\left.\begin{array}{l}\text{Amount to which 1 will accumulate in}\\ \quad\text{the time } x \text{ at the rate of interest } i\end{array}\right\}=y=(1+i)^x \quad . \quad . \quad (1)$$

$$\left.\begin{array}{l}\text{Present value of 1 due } x \text{ years hence at}\\ \quad\text{the rate of interest } i\end{array}\right\}=y=(1+i)^{-x} \quad . \quad . \quad (2)$$



Taking (1), we have (referring to Figure above),

$$\text{Area of figure } OA_1P_1S = \int_0^1 y\,dx$$

$$= \int_0^1 (1+i)^x dx$$

$$= \frac{1+i}{\log_\epsilon(1+i)} - \frac{1}{\log_\epsilon(1+i)}$$

$$= \frac{i}{\log_\epsilon(1+i)} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let us now, instead of taking OS equal to 1, as is assumed in Art. (16), put it equal to $\frac{\log_\epsilon(1+i)}{i}$, and reduce all the other ordinates in the same ratio; then we shall have

$$\text{Area of figure } OA_1P_1S = \frac{\log_\epsilon(1+i)}{i}\int_0^1 y\,dx$$

$$= 1, \text{ from (3)} \quad . \quad . \quad . \quad . \quad (4)$$

Also, $$\text{area of figure } OA_2P_2S = \frac{\log_\epsilon(1+i)}{i}\int_0^2 y\,dx$$

$$= \frac{\log_\epsilon(1+i)}{i} \cdot \frac{(1+i)^2-1}{\log_\epsilon(1+i)}$$

$$= \frac{(1+i)^2-1}{i} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, $$\text{area of figure } OA_3P_3S = \frac{(1+i)^3-1}{i} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$. \quad . \quad . \quad . \quad = \quad . \quad . \quad .$$

Finally, $$\text{area of figure } OA_nP_nS = \frac{(1+i)^n-1}{i} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence we see that, making the above adjustment of the ordinates, the area contained between the initial ordinate, the axis of x, the logarithmic curve, and the ordinate corresponding to the value of x for $x=n$, will denote the amount of an annuity of 1 accumulated for n years.

Similarly, we shall have for present values,

$$\text{Area of figure } SOV_nQ_n = \frac{\log_\epsilon(1+i)}{i}\int_0^n (1+i)^{-x}dx$$

$$= \frac{\log_\epsilon(1+i)}{i}\left\{-\frac{(1+i)^{-n}}{\log_\epsilon(1+i)} + \frac{1}{\log_\epsilon(1+i)}\right\}$$

$$= \frac{1-v^n}{i}$$

$$= \text{Present value of annuity for } n \text{ years} \quad . \quad (8)$$

Again,

$$\text{Area of figure } Q_mV_mV_nQ_n = \frac{\log_\epsilon(1+i)}{i}\int_m^n (1+i)^{-x}dx$$

$$= \frac{\log_\epsilon(1+i)}{i}\left\{-\frac{(1+i)^{-n}}{\log_\epsilon(1+i)} + \frac{(1+i)^{-m}}{\log_\epsilon(1+i)}\right\}$$

$$= \frac{(1+i)^{-m}-(1+i)^{-n}}{i}$$

$$= \frac{v^m - v^n}{i}$$

$$= a_{\overline{n}|} - a_{\overline{m}|}$$

$$= a_{\overline{m+t}|} - a_{\overline{m}|}, \text{ where } t=n-m$$

$$= \left\{\begin{array}{l}\text{Value of an annuity for } t \text{ years} \\ \quad\text{deferred } m \text{ years (Art. 31).}\end{array}\right.$$

Again, if β denote the angle which the tangent drawn to the curve at the point corresponding to ordinates y, x, makes with the axis of x, we have

$$\tan\beta = \frac{dy}{dx}$$

$$= -\frac{(1+i)^{-x}}{\log_\epsilon(1+i)}.$$

Make $x=\infty$, and this becomes

$$\tan\beta=0;$$

from which, since $y=0$ when $x=\infty$, it follows that the axis of x touches the curve at infinity.

$$\left.\begin{array}{l}\text{Thus, area included between the}\\ \quad\text{curve, the axis of } y\text{, and the}\\ \quad\text{point of contact at } x=\infty\end{array}\right\} =\frac{\log_\epsilon(1+i)}{i}\int_0^\infty(1+i)^{-x}dx$$

$$=\frac{\log_\epsilon(1+i)}{i}\left\{0+\frac{1}{\log_\epsilon(1+i)}\right\}$$

$$=\frac{1}{i}$$

$$=\text{Value of a perpetuity }=a_\infty.$$

It should be remarked that in the logarithmic curve the subtangent—that is, the distance from the origin along the axis of x from the ordinate y to the point where the tangent at any point (x, y) cuts the axis of x—is constant. Thus

$$\text{subtangent}=\frac{y}{\tan\beta}$$

$$=\frac{y}{\frac{dy}{dx}}$$

$$=\frac{(1+i)^x}{(1+i)^x\log_\epsilon(1+i)}$$

$$=\frac{1}{\log_\epsilon(1+i)}.$$

It will further be remarked that in the logarithmic curve, whereas the abscissas denoting the time are in arithmetic progression, the ordinates denoting the corresponding amounts or present values are in geometric progression.

If we refer to (2) of the Illustrations of Chapter II., we see that the constant multiplier to obtain the values of annuities payable momently from those payable once a year, is $\frac{i}{\delta}=\frac{i}{\log_\epsilon(1+i)}$, which is the ratio by which the ordinates of the curve denoted by $y=(1+i)^{-x}$ have been reduced, in order for the areas between the curves and the ordinates to denote the values and amounts of annuities payable once a year.

(74).—In this Article it is proposed to give a demonstration of a formula in constant use in the higher mathematics, and which has lately been specially applied by De Morgan, Mr. W. S. B. Woolhouse, and others, in the determination of questions involving the theory of compound interest.

On a formula for summation involving differential coefficients.

Let V denote a function of a variable quantity x, in which the symbol x may be conceived to be the abscissa of a curve and may be employed to represent an interval of time; and suppose the development of V in powers of x to be

$$V_x = V_0 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots .$$

Then, by successive differentiation,

$$\left(\frac{dV}{dx}\right)_x = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \ldots .$$

$$\left(\frac{dV}{dx}\right)_0 = A.$$

$$\left(\frac{d^2V}{dx^2}\right)_x = 2B + 6Cx + 12Dx^2 + 20Ex^3 + \ldots .$$

$$\left(\frac{d^3V}{dx^3}\right)_x = 6C + 24Dx + 60Ex^2 + \ldots .$$

$$\left(\frac{d^3V}{dx^3}\right)_0 = 6C$$

$$\&c. = \&c.$$

$$\therefore \quad V_0 + V_x = 2V_0 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots .$$

$$\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_x = -2Bx - 3Cx^2 - 4Dx^3 - 5Ex^4 - \ldots .$$

$$\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_x = -24Dx - 60Ex^2 - \ldots .$$

$$\&c. = \&c.$$

Also, by integration,

$$\int V dx = V_0 x + A\frac{x^2}{2} + B\frac{x^3}{3} + C\frac{x^4}{4} + D\frac{x^5}{5} + E\frac{x^6}{6} + \ldots .$$

Substituting for A, B, C, &c., their values in terms of the differential coefficients, we obtain

$$\int V dx = \frac{x}{2}\left(V_0 + V_x\right) + \frac{x^2}{12}\left\{\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_x\right\}$$
$$- \frac{x^4}{720}\left\{\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_x\right\} + \ldots.$$

Now suppose the quantity x to pass over successive intervals, each equal to $\frac{1}{m}$, namely, from 0 to $\frac{1}{m}$, $\frac{1}{m}$ to $\frac{2}{m}$, $\frac{2}{m}$ to $\frac{3}{m}$, &c. By applying the formula to each of these intervals, we get

$$\int_0^{\frac{1}{m}} V dx = \frac{1}{2m}\left(V_0 + V_{\frac{1}{m}}\right) + \frac{1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_{\frac{1}{m}}\right\}$$
$$- \frac{1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_{\frac{1}{m}}\right\} + \ldots.$$

$$\int_{\frac{1}{m}}^{\frac{2}{m}} V dx = \frac{1}{2m}\left(V_{\frac{1}{m}} + V_{\frac{2}{m}}\right) + \frac{1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_{\frac{1}{m}} - \left(\frac{dV}{dx}\right)_{\frac{2}{m}}\right\}$$
$$- \frac{1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_{\frac{1}{m}} - \left(\frac{d^3V}{dx^3}\right)_{\frac{2}{m}}\right\} + \ldots.$$

&c. = &c.

$$\int_{\omega-\frac{1}{m}}^{\omega} V dx = \frac{1}{2m}\left(V_{\omega-\frac{1}{m}} + V_\omega\right) + \frac{1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_{\omega-\frac{1}{m}} - \left(\frac{dV}{dx}\right)_\omega\right\}$$
$$- \frac{1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_{\omega-\frac{1}{m}} - \left(\frac{d^3V}{dx^3}\right)_\omega\right\} + \ldots$$

Adding these results, we get

$$\int_0^{\omega} V dx = \frac{1}{m}\left(\tfrac{1}{2}V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} + V_{\frac{3}{m}} + \ldots + V_{\omega-\frac{1}{m}} + \tfrac{1}{2}V_\omega\right)$$
$$+ \frac{1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_\omega\right\} - \frac{1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_\omega\right\} + \ldots$$

Let $\Sigma^{(m)}V$ denote $\frac{1}{m}\left(V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} + \ldots + V_\omega\right)$; then

$$\Sigma^{(m)}V = \int_0^{\omega} V dx + \frac{1}{2m}\left(V_0 + V_\omega\right) - \frac{1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_\omega\right\}$$
$$+ \frac{1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_\omega\right\} - \text{\&c.} \quad . \quad . \quad (1)$$

This formula is ascribed to Euler, and is used by Legendre and Laplace, and others.

If we put $m=1$, so that the values of the function to be summed are taken at consecutive integer or annual intervals, the formula (1) becomes

$$\Sigma^{(1)}V=\int_0^\omega V dx+\frac{1}{2}\left(V_0+V_\omega\right)-\frac{1}{12}\left\{\left(\frac{dV}{dx}\right)_0-\left(\frac{dV}{dx}\right)_\omega\right\}$$
$$+\frac{1}{720}\left\{\left(\frac{d^3V}{dx^3}\right)_0-\left(\frac{d^3V}{dx^3}\right)_\omega\right\}-\&c.$$

and subtracting from (1), we get

$$\Sigma^{(m)}V=\Sigma^{(1)}V-\frac{m-1}{2m}\left(V_0+V_\omega\right)+\frac{m^2-1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0-\left(\frac{dV}{dx}\right)_\omega\right\}$$
$$-\frac{m^4-1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_0-\left(\frac{d^3V}{dx^3}\right)_\omega\right\}+\&c. \quad . \quad . \quad . \quad (2)$$

which is a general formula for deducing the sum of the values of a function, proceeding by intervals of $\frac{1}{m}$, from the sum of the values of the same function, proceeding by intervals of 1.

The demonstration of the formula just given is taken from a paper by Mr. W. S. B. Woolhouse, in vol. xv. of the *Journal of the Institute of Actuaries*, p. 98.

Where the function V_x is such that for $x=\omega$ the terminal values V_ω, $\left(\frac{dV}{dx}\right)_\omega$, $\left(\frac{d^3V}{dx^3}\right)_\omega$, &c., severally become zero, and continue zero for values of x beyond $x=\omega$, the formula (2) is greatly simplified. Such, it may be added, is the case when the formula is applied to annuities and assurances on lives for the whole duration of life; and to these it has been extensively applied (see the paper above referred to, in vol. xv.). Further information on the subject will be found in papers on pp. 61 and 301 of vol. xi., and in vol. xviii., p. 311. On p. 308, vol. xv., the formula (2) has been obtained directly by use of the method of separation of symbols.

Illustration of formula by application to the determination of the value of an annuity at simple interest.

(75).—As an illustration of formula (1) of the preceding Article, we will apply it to find the value of an annuity at simple interest (see Arts. 19 and 20).

Here $V_x=\frac{1}{1+xi}$, and we have, putting $m=1$ in formula (1), and $\omega=n$,

$$\Sigma^{(1)}V=1+\frac{1}{1+i}+\frac{1}{1+2i}+\ldots+\frac{1}{1+ni}$$

$$\int_0^n V\,dx=\int_0^n \frac{1}{1+ix}\,dx=\frac{1}{i}\log_\epsilon(1+ni)$$

$$\frac{1}{2}(V_0+V_n)=\frac{1}{2}\left(1+\frac{1}{1+ni}\right)$$

$$\frac{1}{12}\left\{\left(\frac{dV}{dx}\right)_0-\left(\frac{dV}{dx}\right)_n\right\}=\frac{1}{12}\left\{-\frac{i}{(1+xi)^2_{x=0}}+\frac{i}{(1+xi)^2_{x=n}}\right\}$$

$$=-\frac{i}{12}\left\{1-\frac{1}{(1+ni)^2}\right\}$$

$$\frac{1}{720}\left\{\left(\frac{d^3V}{dx^3}\right)_0-\left(\frac{d^3V}{dx^3}\right)_n\right\}=\frac{1}{720}\left\{-\frac{6i^3}{(1+xi)^4_{x=0}}+\frac{6i^3}{(1+xi)^4_{x=n}}\right\}$$

$$=-\frac{i^3}{120}\left\{1-\frac{1}{(1+ni)^4}\right\}.$$

Hence

$$\frac{1}{1+i}+\frac{1}{1+2i}+\ldots+\frac{1}{1+ni}=\frac{1}{i}\log_\epsilon(1+ni)-\frac{1}{2}\left(1-\frac{1}{1+ni}\right)$$

$$+\frac{i}{12}\left\{1-\frac{1}{(1+ni)^2}\right\}-\frac{i^3}{120}\left\{1-\frac{1}{(1+ni)^4}\right\}+\quad\ldots \quad \text{(A)}$$

This application of the formula (1) is given by De Morgan in the *Journal of the Institute*, vol. xii., p. 251.

If, instead of taking V_0 as the initial value in the formula (1), we take V_1, then it will be found to give the following result:—

$$\frac{1}{1+i}+\frac{1}{1+2i}+\ldots+\frac{1}{1+ni}=V_1+V_2+\ldots+V_n$$

$$=\frac{1}{i}\log_\epsilon\frac{1+ni}{1+i}+\frac{1}{2}(V_1+V_n)$$

$$+\frac{i}{12}(V_1^2-V_n^2)-\frac{i^3}{120}(V_1^4-V_n^4)\text{ approximately} \quad . \quad . \quad \text{(B)}$$

De Morgan, in vol. xiii., p. 143, of the *Journal*, gives this form and tests its accuracy in the case of $i=\cdot 10$ and $n=10$, and finds that whereas the formula gives the value 6·687715, the value found by the use of reciprocals is 6·687714, being a difference of only ·000001.

(76). Another formula, giving $\Sigma^{(m)}V$ in terms of $\Sigma^{(1)}V$ and finite differences instead of differential coefficients, is due, it is believed, to the late Sir John Lubbock.

On Lubbock's formula for summation involving finite differences.

The formula is as follows, using the same notation as for Art. (74):—

$$\Sigma^{(m)}V = \Sigma^{(1)}V - \frac{m-1}{2m}\left(V_0 + V_\omega\right) - \frac{m^2-1}{12m^2}\left(\Delta V_{\omega-1} - \Delta V_0\right)$$

$$- \frac{m^2-1}{24m^2}\left(\Delta^2 V_{\omega-2} + \Delta^2 V_0\right)$$

$$- \frac{(m^2-1)(19m^2-1)}{720m^4}\left(\Delta^3 V_{\omega-3} - \Delta^3 V_0\right)$$

$$\ldots\ldots \quad (3)$$

The next five terms of the series are as follows:—

$$- \frac{(m^2-1)(9m^2-1)}{480m^4}\left(\Delta^4 V_{\omega-4} + \Delta^4 V_0\right)$$

$$- \frac{(m^2-1)(863m^4-145m^2+2)}{60480m^6}\left(\Delta^5 V_{\omega-5} - \Delta^5 V_0\right)$$

$$- \frac{(m^2-1)(275m^4-61m^2+2)}{24192m^6}\left(\Delta^6 V_{\omega-6} + \Delta^6 V_0\right)$$

$$- \frac{(m^2-1)(33953m^6-9247m^4+497m^2-3)}{3628800m^8}\left(\Delta^7 V_{\omega-7} - \Delta^7 V_0\right)$$

$$- \frac{(m^2-1)(8183m^6-2617m^4+197m^2-3)}{1036800m^8}\left(\Delta^8 V_{\omega-8} + \Delta^8 V_0\right).$$

For further information, reference may be made to Mr. Woolhouse's paper, already mentioned, in vol. xi. of the *Journal of the Institute of Actuaries;* also to papers by Mr. T. B. Sprague, in vol. xviii., p. 305, and vol. xxii., p. 55. In the former of Mr. Sprague's papers will be found a method of obtaining the formula different in character from that used by Mr. Woolhouse in vol. xi.; and in the paper in vol. xxii. will be found numerical examples of its application to annuities and assurances. On p. 313 of vol. xviii. is given a table of the values of the coefficients, multiplied throughout by m, as far as those of $\Delta^6 V_{\omega-6} + \Delta^6 V_0$. It should be mentioned that the formula (3) may be applied to obtain formula (2) of Article (74). (See vol. xv., p. 307.)

Numerical illustration of Lubbock's formula by the determination of the value of an annuity-certain.

(77). The following is one of Mr. Sprague's numerical examples:—

Required to find the value of an annuity-certain for 50 years, interest at the rate of 5 per-cent. Putting $m=7$, and taking the values of v, v^8, v^{15}, . . . from Interest Tables, we have

$$\begin{array}{lllllll}
v\ ={\cdot}952381 & & & & & & \\
 & -{\cdot}275542 & & & & & \\
v^{8}={\cdot}676839 & & +{\cdot}079720 & & & & \\
 & -{\cdot}195822 & & -{\cdot}023065 & & & \\
v^{15}={\cdot}481017 & & +{\cdot}056655 & & +{\cdot}006673 & & \\
 & -{\cdot}139167 & & -{\cdot}016392 & & -{\cdot}001929 & \\
v^{22}={\cdot}341850 & & +{\cdot}040263 & & +{\cdot}004744 & & +{\cdot}000554 \\
 & -{\cdot}098904 & & -{\cdot}011648 & & -{\cdot}001375 & \\
v^{29}={\cdot}242946 & & +{\cdot}028615 & & +{\cdot}003369 & & +{\cdot}000402 \\
 & -{\cdot}070289 & & -{\cdot}008279 & & -{\cdot}000973 & \\
v^{36}={\cdot}172657 & & +{\cdot}020336 & & +{\cdot}002396 & & \\
 & -{\cdot}049953 & & -{\cdot}005883 & & & \\
v^{43}={\cdot}122704 & & +{\cdot}014453 & & & & \\
 & -{\cdot}035500 & & & & & \\
v^{50}={\cdot}087204 & & & & & & \\
\hline
\quad 3{\cdot}077598 & & & & & &
\end{array}$$

$$\begin{aligned}
\text{Hence } a_{\overline{50}|} &= 7\times 3{\cdot}077598-3({\cdot}087204+{\cdot}952381)-\tfrac{4}{7}(-{\cdot}035500+{\cdot}275542)-\tfrac{2}{7}({\cdot}014453+{\cdot}079720) \\
&\quad -{\cdot}180758(-{\cdot}005883+{\cdot}023065)-{\cdot}12828({\cdot}002396+{\cdot}006673)-{\cdot}097510(-{\cdot}000973+{\cdot}001929) \\
&\quad -{\cdot}077595(-{\cdot}000402+{\cdot}000554) \\
&= 21{\cdot}543186-3{\cdot}118755=18{\cdot}424431 \\
&\qquad\qquad -\ {\cdot}137167=18{\cdot}287264 \\
&\qquad\qquad -\ {\cdot}026907=18{\cdot}260357 \\
&\qquad\qquad -\ {\cdot}003106=18{\cdot}257251 \\
&\qquad\qquad -\ {\cdot}001163=18{\cdot}256088 \\
&\qquad\qquad -\ {\cdot}000093=18{\cdot}255995 \\
&\qquad\qquad -\ {\cdot}000074=18{\cdot}255921
\end{aligned}$$

The value of $a_{\overline{50}|}$ given by the Interest Tables is 18·255925.

On a formula for the value of $\frac{1}{a_{\overline{n}|}}$ in terms of n and i.

(78). Referring to De Morgan's *Differential and Integral Calculus*, p. 315, we find it shown that

$$\frac{i}{(1+i)^n-1}=\frac{1}{n}-\frac{n-1}{2n}i+\frac{n^2-1}{2.6n}i^2-\frac{n^2-1}{2.3.4n}i^3+\frac{(n^2-1)(19-n^2)}{2.3.4.30n}i^4$$
$$-\frac{(n^2-1)(9-n^2)}{2.3.4.5.4n}i^5+\frac{(n^2-1)(863-145n^2+2n^4)}{2.3.4.5.6.84n}i^6-\,..$$

Now

$$\frac{i}{(1+i)^n-1}+i=\mathrm{P}_{\overline{n}|}+i. \quad \text{(See Art. 26).}$$

$$=\frac{1}{a_{\overline{n}|}}.$$

Hence we get the following series expressing $\frac{1}{a_{\overline{n}|}}$ in terms of i:—

$$\frac{1}{a_{\overline{n}|}}=\frac{1}{n}+\frac{n+1}{2n}i+\frac{n^2-1}{2.6n}i^2-\frac{n^2-1}{2.3.4n}i^3+\frac{(n^2-1)(19-n^2)}{2.3.4.30n}i^4$$
$$-\frac{(n^2-1)(9-n^2)}{2.3.4.5.4n}i^5+\frac{(n^2-1)(863-145n^2+2n^4)}{2.3.4.5.6.84n}i^6-\ldots$$

(79) Referring to (Art. 54) and following Articles, we have, by Taylor's Theorem in the *Differential Calculus*, **Deduction of formulas (D) and (D₁) of Chapter IV.**

$$\phi(x)=\phi(i+\rho)=\phi(i)+\rho\frac{d\phi}{di}+\frac{\rho^2}{2}\frac{d^2\phi}{di^2} \text{ approximately} \quad . \quad . \quad (1)$$

Let, now, $\phi(i)=\dfrac{1-v^n}{i}=a'$ and $\phi(x)=a$;

$$\therefore \quad \frac{d\phi}{di}=-\frac{1-v^n}{i^2}+\frac{nv^{n+1}}{i}=-\frac{1}{i}(a'-nv^{n+1}),$$

$$\text{and } \frac{d^2\phi}{di^2}=-\frac{1}{i}\left\{-\frac{1-v^n}{i^2}+\frac{nv^{n+1}}{i}\right\}+\frac{1-v^n}{i^3}-\frac{nv^{n+1}}{i^2}-\frac{n.\overline{n+1}v^{n+2}}{i}$$

$$=-\frac{1}{i^2}\left\{n.\overline{n+1}.iv^{n+2}-2\left(\frac{1-v^n}{i}-nv^{n+1}\right)\right\},$$

and substituting these results in (1) we have

$$\frac{1}{i^2}\left\{\frac{n.\overline{n+1}iv^{n+2}}{2}-(a'-nv^{n+1})\right\}\rho^2+\frac{1}{i}(a'-nv^{n+1})\rho-(a'-a)=0.$$

Using the method adopted for formula (B) in (Art. 56), taking as the first approximation to the value of ρ, $\rho=\dfrac{i(a'-a)}{a'-nv^{n+1}}$, obtained from $\rho=\dfrac{a-a'}{\dfrac{d\phi}{di}}=-\dfrac{a-a'}{\dfrac{a'-nv^{n+1}}{i}}=\dfrac{i(a'-a)}{a'-nv^{n+1}}$, we get, writing $\rho\times\dfrac{i(a'-a)}{a'-nv^{n+1}}$ for ρ^2 in the equation

$$\frac{n.n+1}{2}v^{n+2}\rho^2+(a-nv^{n+1})\rho-(a'-a)i=0, \text{ (see (6) of Art. 56)}$$

$$\rho=\frac{i(a'-a)}{a-nv^{n+1}+\dfrac{n.n+1}{2}v^{n+2}\dfrac{i(a'-a)}{a'-nv^{n+1}}},$$ which is formula (D) of Art. (58).

The formula obtained from (1) by the use of the method of Art. (56) and adopting the notation of Art. (62), is

$$\rho=\frac{a-a_1}{\dfrac{da_1}{di}+\dfrac{1}{2}\dfrac{a-a_1}{\dfrac{da_1}{di}}\cdot\dfrac{d^2a_1}{di^2}}.$$

If, in this formula, instead of $\frac{da_1}{di}$, $\frac{d^2a_1}{di^2}$, we substitute their approximate values (see Mr. Woolhouse's paper, vol. xi., p. 317, &c.), namely,—

$$\frac{da_1}{di}=a_0=\frac{\Delta a_1-\frac{1}{2}\Delta^2 a_1}{h}$$

$$\frac{d^2a_1}{di^2}=b_0=\frac{\Delta^2 a_1}{h^2},$$

we get the formula (D_1) of Art. (62).

Deduction of Mr. Gray's formula (D) of Art. (47) from Mr. Makeham's formula (A) of Art. (45).

(80) It may be found of some interest to show how Mr. Gray's formula (D) of Art. (47) may be directly obtained from Mr. Makeham's formula (A) of Art. (45).

Mr. Makeham's formula is

$$X=u_1v+u_2v^2+u_3v^3+\ldots+u_nv^n$$

$$=u_1\overset{1}{V}_n+\Delta u_1\overset{2}{V}_n+\Delta^2u_1\overset{3}{V}_n+\ldots$$

$$\therefore\quad iX=u_1(1-v^n)+(\overset{1}{V}-nv^n)\Delta u_1+\left(\overset{2}{V}-\frac{n.n-1}{2}\cdot v^n\right)\Delta^2u_1+\ldots$$

omitting, for convenience, the subscript in V_n, and writing for $\overset{1}{V}$, $\overset{2}{V}$, &c., their values as given in Art. (45).

$$\therefore\ iX=u_1+\overset{1}{V}\Delta u_1+\overset{2}{V}\Delta^2u_1+\,..\,-v^n\left(u_1+n\Delta u_1+\frac{n.n-1}{2}\Delta^2u_1+\,..\right)$$

$$=u_1+\overset{1}{V}\Delta u_1+\overset{2}{V}\Delta^2u_1+\,..\,-v^nu_{n+1}$$

$$=u_1+\frac{1-v^n}{i}\Delta u_1+\frac{\overset{1}{V}-nv^n}{i}\Delta^2u_1+\,\ldots\,-v^nu_{n+1}$$

$$=u_1+\frac{\Delta u_1}{i}+\frac{\Delta^2u_1}{i}\overset{1}{V}+\frac{\Delta^3u_1}{i}\overset{2}{V}+\,\ldots\,-v^nu_{n+1}$$

$$-\frac{v^n}{i}\left(\Delta u_1+n\Delta^2u_1+\frac{n.n-1}{2}\Delta^3u_1+\,..\right)$$

$$=u_1+\frac{\Delta u_1}{i}+\frac{\Delta^2u_1}{i}\overset{1}{V}+\frac{\Delta^3u_1}{i}\overset{2}{V}+\,\ldots\,-v^nu_{n+1}-\frac{v^n}{i}\Delta u_{n+1}$$

$$=u_1+\frac{\Delta u_1}{i}+\frac{\Delta^2u_1}{i^2}(1-v^n)+\frac{\Delta^3u_1}{i^2}(\overset{1}{V}-nv^n)$$

$$+\frac{\Delta^4u_1}{i^2}\left(\overset{2}{V}-\frac{n.n-1}{2}v^n\right)+\,..\,-v^n\left(u_{n+1}+\frac{\Delta u_{n+1}}{i}\right)$$

$$=u_1+\frac{\Delta u_1}{i}+\frac{\Delta^2u_1}{i^2}+\,..\,-\frac{v^n}{i^2}\left(\Delta^2u_1+n\Delta^3u_1+\frac{n.n-1}{2}\Delta^4u_1+\,..\right)$$

$$-v^n\left(u_{n+1}\frac{\Delta u_{n+1}}{i}\right)$$

. . . .

$$=u_1+\frac{\Delta u_1}{i}+\frac{\Delta^2u_1}{i^2}+\,\ldots\,-v^n\left(u_{n+1}+\frac{\Delta u_{n+1}}{i}+\frac{\Delta^2u_{n+1}}{i^2}+\,\ldots\right)$$

$$\therefore\ X=u_1v+u_2v^2+u_3v^3+\,..\,u_nv^n$$

$$=\frac{u_1}{i}+\frac{\Delta u_1}{i^2}+\frac{\Delta^2u_1}{i^3}+\,\ldots\,-v^n\left(\frac{u_{n+1}}{i}+\frac{\Delta u_{n+1}}{i^2}+\frac{\Delta^2u_{n+1}}{i^3}+\,\ldots\right)$$

which is Mr. Gray's formula (D) of Art. (47).

On a general formula for the *Present Values* of the payments of Interest and repayments of Capital, respectively, in the case of a given annuity.

(81) Referring to Art. (53), it is there shown that if

$$X=u_1v+u_2v^2+u_3v^3+\,..\,+u_nv^n \quad . \quad . \quad . \quad . \quad (1)$$

then the *present value* of the payments on account of repayment of capital is

$$X-iv^2(u_1+2u_2v+3u_3v^2+\,..\,+nu_nv^{n-1}) \quad . \quad . \quad . \quad (2)$$

Now from (1) we have, by differentiation,

$$\frac{dX}{di}=-v^2(u_1+2u_2v+3u_3v^2+\,..\,+nu_nv^{n-1}) \quad . \quad . \quad (3)$$

Hence from (2) we have

$$\left.\begin{array}{l}\textit{Present value} \text{ of payments} \\ \quad\text{on account of repayment} \\ \quad\text{of capital}\end{array}\right\} = X - \left(-i\frac{dX}{di}\right)$$

$$= X + i\frac{dX}{di}$$

$$= \frac{d}{di}(iX) \quad . \quad . \quad . \quad . \quad \text{(A)}$$

and

$$\left.\begin{array}{l}\textit{Present value} \text{ of payments} \\ \quad\text{on account of interest}\end{array}\right\} = -i\frac{dX}{di} \quad . \quad . \quad . \quad . \quad \text{(B)}$$

It is possible that these general results may be found, on further investigation, to be of some service in questions involving annuities-certain.

The result arrived at in (A) has been already obtained in the simple case of an ordinary annuity.

Thus if $$X = \frac{1-v^n}{i}$$

$$\therefore \quad iX = 1 - v^n$$

$$\therefore \quad \frac{d}{di}(iX) = nv^{n+1}$$

which has been already independently obtained in Art. (69).

CHAPTER VII.

ON INTEREST TABLES.

On the calculation of Interest Tables

(82) As will be seen further on in this chapter, various sets of interest tables have from time to time been computed and published, by the aid of which the solution of interest questions is greatly facilitated. Of course, in any set of tables, the greater the number of functions involving i and n (the rate of interest and the time), whose values are tabulated, and the more extensive the range of values of i and n for which the functions are tabulated, the more complete would such tables be. Apart, however, from the enormous labour involved in the preparation of anything approaching to a complete set of Interest Tables, there is the inevitable tendency towards bulkiness, which, setting aside the cost of production, practically restricts the computation of such tables within well-defined limits.

On the various functions usuall tabulated.

(83) A reference to the detailed examination of some of the existing tables, which will be found further on, will serve to show that (dismissing simple interest) the functions usually tabulated are—

$$(1+i)^n,\ (1+i)^{-n},\ \frac{(1+i)^n-1}{i},\ \frac{1-(1+i)^{-n}}{i},\ \left\{\frac{1-(1+i)^{-n}}{i}\right\}^{-1},$$
$$\left\{\frac{(1+i)^n-1}{i}\right\}^{-1}.$$

Referring, however, to Art. (26) we know that the difference between the last two functions is always i, so that it would appear to be unnecessary to tabulate both; although, from the fact that

some authors have tabulated both, it would appear to be not improbable that the connection between them has been overlooked.

For certain purposes, moreover, it has been found desirable to tabulate certain modified forms of some of the above functions.

Thus, for instance, Art. (26), we know that

$$\frac{1-(1+i)^{-n}}{i}=\frac{1}{P_{\overline{n|}}+i},$$

which assumes that the entire capital bears interest at the rate i for the term of n years, and that the sinking fund for accumulation, namely, $P_{\overline{n|}}$, bears interest at the same rate i. As is shown in Art. (27), and illustrated in examples 6 and 7 of Chapter IV. given in Chapter V., it may be necessary to use a modification of this formula, namely, $\frac{1}{P_{\overline{n|}}+i'}$, where i' denotes the rate of interest the original capital is to bear for the entire period of n years, and $P_{\overline{n|}}$, the sinking fund for accumulation, is to bear interest at a different rate i. Accordingly, we find that at least one set of tables of the values of the function $\frac{1}{P_{\overline{n|}}+i'}$ has been computed and published. (See *Hoskold*, Art. (87).*

Reference to the detailed examination of various tables will show that other functions have been tabulated for special purposes.

Thus Lieut.-Col. Oakes has tabulated

$$1-p=\left(\frac{i'}{2}-\frac{i}{2}\right)\frac{1-\left(1+\frac{i'}{2}\right)^{-n}}{\frac{i'}{2}},$$

Mr. Herbert Johnson has tabulated

$$p=(i'-i)\frac{1}{P_{\overline{n|}}+i'} \text{ where } P_{\overline{n|}}=\left\{\frac{(1+x)^n-1}{x}\right\}^{-1}.$$

Here three rates of interest, i, i', and x, are involved.

On various methods of calculating and checking a series of tabulated values of certain functions.

(84) It is not proposed to enter into any lengthy description of the various methods which may be adopted for the purpose of calculating any series of values of any one of the above-mentioned functions, and of checking the results by some independent process. For information on this subject reference may be made to a most

* Other Tables have also been computed and published. For instance, Tables of this description will be found on p. 8, of vol. i., of the *Journal of the Institute* (part 2), by the late Peter Hardy, and in W. D. Biden's *Tables*, published in 1864.

valuable work by Mr. Peter Gray, entitled *Tables and Formulæ for the Computation of Life Contingencies, &c.* Laytons, 1870.

It will be sufficient here to give some general explanations.

(*a*) Taking the function $(1+i)^n$, we may commence with $1+i$, and then multiplication by itself would give $(1+i)^2$, and this result again multiplied by $1+i$, would give $(1+i)^3$, and so on. Where i is ·02, ·03, &c., these multiplications can be readily made, and to any number of decimal places, but it will be proper always to take two more places than are ultimately wanted.

The results should be verified from time to time either by the use of logarithms, or thus:—

$$
\begin{array}{lcl}
(1+i)^2 \times (1+i)^3 & \text{should give} & (1+i)^5, \\
(1+i)^5 \times (1+i)^5 & \text{"} & (1+i)^{10}, \\
(1+i)^{10} \times (1+i)^{10} & \text{"} & (1+i)^{20}, \\
(1+i)^{20} \times (1+i)^{10} & \text{"} & (1+i)^{30},
\end{array}
$$

and so on.

(*b*) The function $(1+i)^{-n}$ might be constructed either like that for $(1+i)^n$, or by continual division of $(1+i)^{-1}$ by $1+i$, or by calculating the last value, say $(1+i)^{-100}$, and then working backwards with the constant multiplier $1+i$.

Thus
$$(1+i)^{-99}=(1+i)(1+i)^{-100},$$
$$(1+i)^{-98}=(1+i)(1+i)^{-99}.$$

Where $(1+i)^n$ is already calculated, the intermediate results can be checked thus:—

$$(1+i)^{-90}=(1+i)^{10}(1+i)^{-100},$$
$$(1+i)^{-80}=(1+i)^{10}(1+i)^{-90}.$$

The final result $(1+i)^0=1=(1+i)(1+i)^{-1}$ will serve as a final check.

(*c*) When $(1+i)^n$ and $(1+i)^{-n}$ have been calculated, then we see that

$$\frac{(1+i)^n-1}{i}=1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}$$

and

$$\frac{1-(1+i)^{-n}}{i}=(1+i)^{-1}+(1+i)^{-2}+\ldots+(1+i)^{-n}$$

are at once obtained by summation.

(*d*) When $(1+i)^n$ and $(1+i)^{-n}$ have not been calculated, then we have

$$\frac{(1+i)^n-1}{i}=(1+i)\frac{(1+i)^{n-1}-1}{i}+1 \quad . \quad . \quad . \quad (1)$$

$$\frac{1-(1+i)^{-(n-1)}}{i}=(1+i)\frac{1-(1+i)^{-n}}{i}-1 \quad . \quad . \quad . \quad (2)$$

In (1) a commencement would be made with $n=1$, and then the values for $n=2$, 3, 4, would be found in the manner indicated.

In (2) a commencement would be made with the last value of n, required say, $n=100$, this value being calculated independently by logarithms.

For examples, reference should be made to Mr. Gray's book, where it is pointed out that, in computing by formulas (1) and (2), the values of $(1+i)^n$ and $(1+i)^{-n}$ are at the same time obtained.

The results may be checked in a variety of ways, either by independent calculation of values at given intervals, or by differencing in the case of (*c*) and comparing with the values of $(1+i)^n$ and $(1+i)^{-n}$, or by summation of the values, in which case we should have

$$\Sigma_1^n\frac{(1+i)^n-1}{i}=\frac{(1+i)-1}{i}+\frac{(1+i)^2-1}{i}+\ldots+\frac{(1+i)^n-1}{i}$$

$$=\frac{(1+i)}{i}\{1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}\}-\frac{n}{i}$$

$$=\frac{1+i}{i}\cdot\frac{(1+i)^n-1}{i}-\frac{n}{i} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\Sigma_1^n\frac{1-(1+i)^{-n}}{i}=\frac{1-(1+i)^{-1}}{i}+\frac{1-(1+i)^{-2}}{i}+\ldots+\frac{1-(1+i)^{-n}}{i}$$

$$=\frac{n}{i}-\frac{1}{i}\{(1+i)^{-1}+(1+i)^{-2}+\ldots+(1+i)^{-n}\}$$

$$=\frac{n}{i}-\frac{1}{i}\cdot\frac{1-(1+i)^{-n}}{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

so that in each case the sum of the entire series of values may be obtained in terms of the last one.

It should be noted that (4) has already been obtained in Art (65).

(85) Mr. Gray has given, in his *Tables and Formulæ*,* various methods of calculating the values of the functions commonly tabulated, but it is doubtful if these are of common use. It is probable that for ordinary purposes the simplest plan is to use ordinary logarithms throughout. Thus, starting with $\log(1+i)$, we get by successive additions $\log(1+i)^2$, $\log(1+i)^3 \ldots$, and subtraction of these results from zero gives $\log(1+i)^{-1}$, $\log(1+i)^{-2}, \ldots$. The numbers corresponding to these logarithms give the series of values $1+i$, $(1+i)^2$, &c., and $(1+i)^{-1}$, $(1+i)^{-2}$, respectively, and these again when summed give the values of $\frac{(1+i)^n-1}{i}$ and $\frac{1-(1+i)^{-n}}{i}$, respectively, for the various values of n. **General observations on the calculation of Interest Tables.**

Care, of course, must be taken that the logarithm of $1+i$ is taken to two or three places more than that to which the results are required, in consequence of an error in the last place being constantly increased as n increases, and also in order to ensure accuracy in taking out the numbers corresponding to the logarithms.

There will be found appended to this Chapter an extensive table of values of $\log_{10}(1+i)$, which has been specially constructed by Mr. Peter Gray for this work.

(86) Various instruments have been from time to time devised for facilitating numerical calculations, the most important and well-known is the Arithmometer of M. Thomas (de Colmar). In a paper by Major-General J. C. Hannyngton, in p. 244 of vol xvi. of the *Journal of the Institute of Actuaries*, will be found an excellent description of the Arithmometer, with examples illustrating its application; and in the following pages of the same volume will be found further information on the same subject. Mr. Peter Gray, on p. 249 of vol. xvii., and pp. 20 and 123 of vol. xviii., has also given detailed illustrations of the working of the Arithmometer. Appended to this Chapter will be found a Note, by Major-General J. C. Hannyngton, on Instrumental Calculations, which, on account of the knowledge and experience of the author, cannot fail to be of interest. **On Instrumental Calculations.**

* In this work are given two valuable tables. In the first, $\log x$ is the argument, and $\log(1+x)$ the tabular result. In the second, $\log x$ is the argument, and $\log(1-x)$ the tabular result. Mr. Gray has also given illustrations of the application of these tables to the calculation of the values and amounts of annuities-certain.

Analytical description of various well-known or readily obtainable Interest Tables.

(87) In this Article, it is proposed to give a somewhat minute description of some of the Interest Tables which have been published. The list is in no sense to be considered as complete, but rather as including only some of the more well-known or readily obtainable tables.

FRANCIS CORBAUX.—"Doctrine of Compound Interest." 1825.

In these tables the functions tabulated are the same as those of Tables III. to XVII. of Mr. Turnbull's Tables, and in addition the values of

$$\left\{\frac{(1+i)^n-1}{i}\right\}^{-1},\ \left\{\frac{\left(1+\frac{i}{2}\right)^{2n}-1}{i}\right\}^{-1} \text{ and } \left\{\frac{\left(1+\frac{i}{4}\right)^{4n}-1}{i}\right\}^{-1};$$

but these would appear to be unnecessary, as they can be respectively obtained from VII., XII., and XVII., by subtracting i. (See Art. 26.)

The values of i are from 3 to 6 per-cent, proceeding by $\frac{1}{4}$th per-cent.

The values of n are from 1 to 100, proceeding by $\frac{1}{2}$ and $\frac{1}{4}$ up to 16, and then by 1. The results are generally to 7 decimal places, but are curtailed for higher values of n to 6 and 5 places.

It is to be noted with regard to these tables, that all the values for the same value of i are given in the same opening of the book.

PETER HARDY, F.R.S.—"Doctrine of Simple and Compound Interest." 1839.

Functions tabulated:

$$(1+i)^n,\ (1+i)^{-n},\ \frac{(1+i)^n-1}{i},\ \frac{1-(1+i)^{-n}}{i},\ \log(1+i)^{-n}.$$

For all values of n from 1 to 100, and for values of i from $\frac{1}{4}$ per-cent to 5 per-cent, proceeding by $\frac{1}{4}$ per-cent; and then for 6, 7, and 8 per-cent.

In the first four functions, the number of decimal places is 4, and 7 in the last.

H. D. HOSKOLD.—"The (Mining) Engineer's Valuing Assistant." 1877.

Table I. gives values of $(1+i)^n$ for values of n from 1 to 100, and for values of i from $\frac{1}{2}$ per-cent to 3 per-cent proceeding by $\frac{1}{4}$th per-cent.

"	3	"	6	"	"	$\frac{1}{2}$	"
"	6	"	25	"	"	1	"

As far as 9 per-cent the number of decimal places is 10, thence to 15 per-cent 6 places, and afterwards 5 places.

Table II. gives values of $\left(1+\frac{i}{2}\right)^{2n}$ where n is from 1 to 50
" " $\left(1+\frac{i}{4}\right)^{4n}$ " 1 to 25 } and $i = {\cdot}03$.

Table III. gives values of $\frac{(1+i)^n-1}{i}$ where n is from 1 to 100, and i from $\frac{1}{2}$ per-cent to 10 per-cent, proceeding as in Table I. The number of decimal places is 10 up to 7 per-cent, and 6 afterwards.

H. D. Hoskold—(*continued*).

Table IV. gives values of $(1+i)^{-n}$ where n is from 1 to 100 as far as 10 per-cent, afterwards from 1 to 50, and i is from 3 to 25 per-cent, proceeding by ½ per-cent as far as 5 per-cent, by 1 per-cent thence to 25 per-cent.

The number of decimal places is 8 throughout.

Table V. gives values of $P_{\overline{n}|}$ or $\left\{\frac{(1+i)^n-1}{i}\right\}^{-1}$, where i is from 1½ per-cent to 20 per-cent, proceeding thus: 1½, 2, 2½, 3, 3¼, 3½, 4, 4¼, 4½, 5, 10, 12, 15, 18, 20. Up to 5 per-cent n is from 1 to 100, and afterwards 1 to 50.

The number of decimal places is 10.

The same table gives values of $P^{(2)}_{\overline{n}|}$ and $P^{(4)}_{\overline{n}|}$, that is, $\left\{\frac{\left(1+\frac{i}{2}\right)^{2n}-1}{i}\right\}^{-1}$ and $\left\{\frac{\left(1+\frac{i}{4}\right)^{4n}-1}{i}\right\}^{-1}$, where n is from 1 to 50 or 25, and i from 3 to 5 per-cent, proceeding by ¼th per-cent, the number of decimal places being 6.

Tables VI., VII., VIII., IX., give values of $\frac{1}{P_{\overline{n}|}+i'}$, where $P_{\overline{n}|}=\left\{\frac{(1+i)^n-1}{i}\right\}^{-1}$, as follows:—

	n	$100i$	$100i'$
Table VI.	1 to 100	2½	3 to 6 by ½, 6 to 25 by 1
„ VII.	„	3	The same as VI.
„ VIII.	„	3½	4, 5, 6, 8, 10, 12, 15, 18, 20, 25
„ IX.	„	4	5, 6, 8, 10, 12, 15, 16, 18, 20, 25

throughout to 8 places of decimals.

Table X. gives values of $\frac{v'^t}{P_{\overline{n}|}+i'}$ where $v'=\frac{1}{1+i'}$, and $P_{\overline{n}|}$ is as in preceding tables.

t is from 1 to 10, i is 3 per-cent.

n , 1 to 100, i' is 4, 5, 6, 8, 10, 12, 15, 18, or 20 per-cent.

Number of decimal places, 6.

Table XI. is similar to Table X., i being 3½ or 4 per-cent, and i' being 20 per-cent.

Table XIII. gives values of $\frac{1-(1+i)^{-n}}{i}$ where i is 2, 2½, 3, or 3½ per-cent, and n is from 1 to 100. Number of decimal places, 5.

Table XVI. gives values of $\frac{v^t}{i}$ where $v=\frac{1}{1+i}$, and t is from 1 to 100; and i is 3, 3½, 4, 4½, 5, 6, 7, or 8 per-cent. Number of decimal places, 4.

CHARLES INGALL.—"Tables for ascertaining the value of Debentures issued, bearing interest at from $3\frac{1}{2}$ to 6 per-cent, and to run from 1 to 7 years by quarters of a year, to pay the buyer from $3\frac{1}{2}$ to 7 per-cent, computed for every $\frac{1}{4}$ per-cent." 1862.

The results are given in £. *s.* *d.*, and are computed from the formula

$$\begin{aligned} 1-p &= \frac{(i'-i)n}{1+ni'} \\ &= \left(\frac{1}{1+ni} - \frac{1}{1+ni'}\right)(1+ni) \\ &= 1 - \frac{1+ni}{1+ni'}, \end{aligned}$$

where n denotes the time before redeemable at par, i the rate of interest which the debenture bears, p the discount at which it is bought, and i' the rate of interest yielded to the purchaser.

W. INWOOD.—"Tables for the purchasing of Estates, &c." 1870. (19th edition.)

Contains the five Compound Interest Tables extracted from John Smart's Tables, but reduced to 4 decimal places. (See *Smart's Tables.*)

It also contains extracts from Thoman's Logarithmic Tables. (See *Thoman's Tables.*)

HERBERT JOHNSON.—"Investment Tables for Stocks and Perpetual and Terminable Debentures." 1881.

Tables I. and II. do not call for notice.

Table III. gives the values of the function

$$p=(i'-i)\frac{1}{P_{\overline{n}|}+i'}\left\{\text{where } P_{\overline{n}|}=\left(\frac{(1+x)^n-1}{x}\right)^{-1}\right\},$$

where p, i, and n are the arguments, and i' the tabular result.

The values of p vary from 35 per-cent discount (that is $-$) to 160 premium (that is $+$).

The values of i vary from 4 to 6 per-cent by $\frac{1}{4}$ per-cent.

The values of n vary from $1\frac{1}{4}$ to 50 years, proceeding by $\frac{1}{4}$ up to 15 years

„	„	„	1 from 15 to 26 years
„	„	„	2 „ 26 „ 36 „
„	„	„	3 „ 36 „ 42 „
„	„	„	4 „ 42 „ 50 „

On the left-hand page x, that is, the rate of interest for reinvestment, is taken at 3 per-cent, and on the right-hand page at 4 per-cent.

Table IV. is similar to Table III.

The number of years proceeds from $1\frac{1}{2}$ to 10 by $\frac{1}{2}$ (year).

The values of p are from 11 per-cent discount ($-$) to 4 per-cent premium ($+$).

The value of i is $3\frac{1}{2}$ per-cent throughout, as is also the value of x, the rate of interest for reinvestment.

In both tables the value of i' is given in £ *s.* *d.* per-cent.

JOHN LAURIE.—"Tables of Simple and Compound Interest, &c." 1776.

No. of Table.	Functions tabulated.	General Remarks.
I.	$(1+i)^n$ and $1+ni$	* In these cases, corresponding values are also given at simple interest.
II.	$(1+i)^{-n}$ and $(1+ni)^{-1}$	
III.	$\frac{(1+i)^n-1}{i}$ *	
IV.	$\frac{1-(1+i)^{-n}}{i}$ *	The rates of interest for which the tables are computed are—3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, and 5 per-cent.
V.	$\left\{\frac{1-(1+i)^{-n}}{i}\right\}^{-1}$ *	
VI.	$\left\{\frac{(1+i)^n-1}{i}\right\}^{-1}=P_{\overline{n}\vert}$ *	The number of years is generally 1 to 50; but in the half-yearly cases generally from $\frac{1}{2}$ to 44, proceeding by $\frac{1}{2}$ years.
VII.	$\{1+a_{\overline{n-1}\vert}\}^{-1}$, where $a_{\overline{n-1}\vert}=\frac{1-(1+i)^{-(n-1)}}{i}$	
VIII.	$\left(1+\frac{i}{2}\right)^{2n}$	
IX.	$\left(1+\frac{i}{2}\right)^{-2n}$	
X.	$\frac{\left(1+\frac{i}{2}\right)^{2n}-1}{i}$ *	In VI., the argument is $P_{\overline{n}\vert}$ for various values from 15 per-cent downwards, and n is the tabular result, and a somewhat similar arrangement in XIII.
XI.	$\frac{1-\left(1+\frac{i}{2}\right)^{-2n}}{i}$ *	
XII.	$\left\{\frac{1-\left(1+\frac{i}{2}\right)^{-2n}}{i}\right\}^{-1}\frac{1}{2}$ *	
XIII.	$\left\{\frac{\left(1+\frac{i}{2}\right)^{2n}-1}{i}\right\}^{-1}\frac{1}{2}$ *	The results are generally given to 7 decimal places, in some cases 6, and are also given in £. s. d.
XIV.	$\{\frac{1}{2}+a^{(2)}_{\overline{n-\frac{1}{2}}\vert}\}^{-1}$	

D. J. McG. McKENZIE.—Two tables on pp. 183, 184 of the *Journal of the Institute of Actuaries*, vol. xxiii.

Table I. gives the values of $\log \dfrac{i}{(1+i)^{\frac{1}{n}}-1}$.

Table II. gives the value of $(1+i)^{\frac{1}{n}}-1$.

The first table is the table referred to on page 116.

The second table gives the rates of interest per interval of $\frac{1}{n}$th of a year, equivalent to an effective annual rate i, except for $n=\infty$, in which case the nominal annual rate is given.

D. J. McG. McKenzie—(*continued*).

In each table i runs from $2\frac{1}{2}$ per-cent to 10, proceeding by $\frac{1}{8}$th per-cent.

In Table I. the number of decimal places is 7.

In Table II. the number of decimal places runs from 8 to 10.

The values of n are 2, 4, 12, 26, 52, and ∞.

Lieut.-Col. W. H. Oakes.—"Loans payable by Drawings and Debenture Interest Tables." 1870.

Function tabulated: $1-p=\left(\frac{i'}{2}-\frac{i}{2}\right)\frac{1-\left(1+\frac{i'}{2}\right)^{-n}}{\frac{i'}{2}}$.

For all values of n from 1 to 60 (half-years), and for values of $100 \times \frac{i}{2}$ from $1\frac{1}{2}$ to $4\frac{1}{2}$ proceeding by $\frac{1}{8}$, and for values of $1-p$ from 99 to 70 proceeding by 1.

$100 \times \frac{i}{2}$ and n (half-years), and $1-p$ are the arguments, and $100 \times \frac{i'}{2}$ the tabular result, which is given to 3 decimal places.

Lieut.-Col. W. H. Oakes.—"Tables of Compound Interest." 1877.

Functions tabulated: $(1+i)^n$, $(1+i)^{-n}$, $\frac{(1+i)^n-1}{i}$, $\frac{1-(1+i)^{-n}}{i}$.

For all values of n from 1 to 100, and for values of i from $\frac{3}{4}$ per-cent to 10 per-cent proceeding by $\frac{1}{8}$ths per-cent.

Number of decimal places, 5 throughout.

Thomas George Rance.—"Compound Interest Tables." 1876.

Functions tabulated: $(1+i)^n$, $(1+i)^{-n}$, $\frac{(1+i)^n-1}{i}$, $\frac{1-(1+i)^{-n}}{i}$.

For all values of n from 1 to 100, and for values of i from $\frac{1}{4}$ per-cent to 10 per-cent, proceeding by $\frac{1}{4}$ per-cent. Number of decimal places, 7 throughout.

John Smart.—"Tables of Interest, Discount, Annuities, &c." 1726.

Functions tabulated: ni, $\frac{ni}{1+ni}$, $(1+i)^n$, $(1+i)^{-n}$, $\frac{(1+i)^n-1}{i}$, $\frac{1-(1+i)^{-n}}{i}$, $\left\{\frac{1-(1+i)^{-n}}{i}\right\}^{-1}$.

In the function ni, n is taken for each day from 1 to 365, and from 1 to 25 years.

" $\frac{ni}{1+ni}$, n is taken for each day from 1 to 365.

" $(1+i)^n$, n is taken for each day from 1 to 365, and every half-year up to 100 years.

In the remaining functions, n is taken for every half-year up to 100 years.

The values of i are in all the tables 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, 6, 7, 8, 9, 10 per-cent. The number of decimal places is throughout 8.

Note.—If n be an odd number of half-years, say $n=t+\frac{1}{2}$, then in the last five functions, the values tabulated are those obtained by substituting $t+\frac{1}{2}$ in the respective formulas for n. (See Art. 30.)*

* These tables will also be found appended to Baily's *Doctrine of Interest and Annuities*.

T. K. STUBBINS.—"Annuity Tables for Building Societies and General Use." 1881.

Table I. gives the values of $\dfrac{1-\left(1+\dfrac{i}{12}\right)^{-n}}{\dfrac{i}{12}}$.

Table II. " " $\dfrac{1-\left(1+\dfrac{i}{4}\right)^{-n}}{\dfrac{i}{4}}$.

Table III. " " $\dfrac{1-\left(1+\dfrac{i}{2}\right)^{-n}}{\dfrac{i}{2}}$.

The values of i are from 3 to 8 per-cent, proceeding by $\frac{1}{2}$ per-cent.

The values of n are, in I., from 1 to 300, proceeding by 1.

" " II., " 1 to 200, "

" " III., " 1 to 100, "

In I. to 3, and in II. and III., to 4 decimal places.

FÉDOR THOMAN.—"Translation of his work on Theory of Compound Interest and Annuities." 3rd edition. 1877.

No. of Table.	Function tabulated.	n	$100i$	No. of Decimal Places.
I. (A)	$\log_{10}(1+i)^n$	From 1 to 100	$\frac{1}{2}$, 1, $1\frac{1}{2}$, and then by $\frac{1}{8}$ths up to 6; 6 to 7 by $\frac{1}{4}$; 7, $7\frac{1}{2}$, 8, 9, 10, 12	7
" (B)	$\log \dfrac{1}{a_{\overline{n}\rceil}}$	Do.		
II.	$\log_{10}(1+i)^{\frac{n}{12}}$	1, 2, 3 .. up to 12	Same as for I.	7
III. (A)	$\log_{10} i$, and $\log_{10}(1+i)$	. . .	Same as for I.	10
" "	$\log^2{}_{10}(1+i)$	. . .	Same as for I.	7
" (B)	$\log_{10} i$, and $\log_{10}(1+i)$	. . .	0 to 10, proceeding by $\frac{1}{10}$th	7 10
" (C)	Same as III. (B)	. . .	0 to 10, proceeding by $\frac{1}{12}$th	Same as III. (B)
IV.	$\dfrac{\dfrac{i}{n}}{(1+i)^{\frac{1}{n}}-1}$	2, 4, 6 or 12	$\frac{1}{4}$ to $1\frac{1}{2}$ by $\frac{1}{4}$ths; $1\frac{1}{2}$ to 6 by $\frac{1}{8}$ths; remainder same as I.	7
	$\dfrac{\left(1+\dfrac{i}{n}\right)^n-1}{i}$	2, or 4		

ANDREW HUGH TURNBULL.—"Tables of Compound Interest and Annuities." 1863.

The following is a summary of the principal tables contained in this work:

No. of Table.	Function tabulated.	n	$100i$
III.	$(1+i)^n$	From 1 to 80	3, 3½, 4, 4½, 5, 6
IV.	$(1+i)^{-n}$	Do.	Do.
V.	$\frac{(1+i)^n-1}{i}$	Do.	Do.
VI.	$\frac{1-(1+i)^{-n}}{i}$	Do.	Do.
VII.	$\left\{\frac{1-(1+i)^{-n}}{i}\right\}^{-1}$	Do.	Do.
VIII.	$\left(1+\frac{i}{2}\right)^{2n}$	From ½ to 40 by ½ (year)	3, 3¼, 3½, 3¾, 4, 4¼, 4½, 4¾, 5, 5½
IX.	$\left(1+\frac{i}{2}\right)^{-2n}$	Do.	Do.
X.	$\frac{\left(1+\frac{i}{2}\right)^{2n}-1}{i}$	Do.	Do.
XI.	$\frac{1-\left(1+\frac{i}{2}\right)^{-2n}}{i}$	Do.	Do.
XII.	$\left\{\frac{1-\left(1+\frac{i}{2}\right)^{-2n}}{i}\right\}^{-1}$	Do.	Do.
XIII.	$\left(1+\frac{i}{4}\right)^{4n}$	From ¼ to 20 by ¼ (year)	3, 3½, 4, 4½, 5, 5½
XIV.	$\left(1+\frac{i}{4}\right)^{-4n}$	Do.	Do.
XV.	$\frac{\left(1+\frac{i}{4}\right)^{4n}-1}{i}$	Do.	Do.
XVI.	$\frac{1-\left(1+\frac{i}{4}\right)^{-4n}}{i}$	Do.	Do.
XVII.	$\left\{\frac{1-\left(1+\frac{i}{4}\right)^{-4n}}{i}\right\}^{-1}$	Do.	Do.
XVIII.	$\frac{B}{i}$	B = 2½, 3, 3½, 4, 5, 6, 7, 8, 9, 10 per-cent	2½ to 6 by ⅛th, 6 to 9 by ¼ and 10
XIX.	$\frac{1}{i}$	. . .	2 to 8 by ⅛th, 8 to 11 by ¼, 11½, 12, 12½

The values are throughout to 7 decimal places, and are also given in £ *s*. *d*.

P. A. VIOLEINE.—"Nouvelles Tables pour les calculs d'Intérêts Simple et Composés, &c." Deuxième édition. 1854.

The following is a summary of these tables as far as they call for notice:—

No. of Table.	Function tabulated.	n	$100i$
V.	$\frac{in}{360}$	From 1 to 30	1 to 10 proceeding by $\frac{1}{8}$ths and $\frac{1}{6}$ths up to 6, and then by $\frac{1}{4}$th and $\frac{1}{3}$rd.
VI.	$\frac{in}{12}$	1 to 12	Do.
VII.	in	1 to 10	Do.
VIII.	$n\left(1+\frac{n-1}{2}i\right)$	Do.	Do.
IX.	$\frac{1+ni}{n\left(1+\frac{n-1}{2}i\right)}$	Do.	Do.
XIII.	$(1+i)^n$	1 to 100	Do.
XIV.	$(1+i)^{\frac{n}{12}}$	1 to 12	Do.
XV.	$(1+i)^{\frac{n}{360}}$	1 to 30	Do.
XVI.	$\frac{(1+i)^n-1}{i}$	1 to 100	Do.
XVII.	$\frac{(1+i)^n-1}{i}$	1 to 12	$\frac{3}{24}$ up to $\frac{12}{24}$ by $\frac{1}{24}$th; $\frac{14}{24}$, $\frac{15}{24}$, $\frac{16}{24}$, $\frac{18}{24}$, $\frac{20}{24}$, $\frac{21}{24}$, $\frac{22}{24}$.
XVIII.	$\frac{(1+i)^n-1}{i}(1+i)$	Do.	Do.
XIX.	Giving n as tabular result in $(1+i)^n=2$		Do.
XX.	$\left\{\frac{1-(1+i)^{-n}}{i}\right\}^{-1}$	1 to 100	Do.
XXI.	Giving $n=\frac{\log\left(1+\frac{i}{P}\right)}{\log(1+i)}$ for various values of P from 0 to 3 per-cent, proceeding by $\frac{1}{16}$ths or other subdivisions		Do.

The compound interest tables are generally given to 8 places of decimals.

Various Interest Tables have, from time to time, been published as part of works on subjects involving the application of the theory of interest. Among others are the following:—

GRIFFITH DAVIES.—*Treatise on Annuities.*

JOHN MILNE.—*Treatise on Annuities and Assurances.* 1815.

DAVID JONES.—*On the Value of Annuities.* 1843.

DAVID CHISHOLM.—*Commutation Tables.* 1858.

NOTE ON INSTRUMENTAL CALCULATIONS,

BY MAJOR-GENERAL J. C. HANNYNGTON.

The duties of an actuary often require him to make laborious calculations, on the correctness of which important conclusions may depend. To relieve mental strain, to avoid error, and to save time, he will do well to avail himself of mechanical assistance.

Foremost among machines stands the Arithmometer of M. Thomas (de Colmar), which has lately been materially improved and simplified by Mr. S. Tate (of No. 6, Waterloo Place, Clerkenwell Close). As now constructed, it is very strong, and may be worked, without risk, at high speed.*

The uses of this machine are endless. The general principle of its operations has been well explained by Mr. Peter Gray in the *Journal of the Institute.*

Next come the various instruments that have been devised for indicating logarithms by measurement. These instruments, whatever be their form, are named sliding rules. As the arithmometer commands that branch of arithmetic which depends on addition and subtraction, so does a good sliding rule command all logarithmic operations.

Professor George Fuller, C.E., of Queen's College, Belfast, has invented an instrument, in which a logarithmic scale winds in a spiral around a revolving cylinder, to which indices are applied. By this means a scale of fifty feet in length is made perfectly manageable. In all parts, four figures are determinable; and, for a large portion, the fifth figure may be correctly estimated.

Mr. Thomas Dixon, of Buttershaw (near Bradford), has invented a flat spiral, or scroll, on which the scale runs from the centre outwards, by which arrangement the graduation is, or might be, in equal divisions throughout. The advantage thus gained is very great. The reading is effected by three or more concentric arms, to be clamped in position as required.

* C. & E. Layton, the publishers, state that there is another form of the machine, known as Elliott's, which is of English make.

A seemingly unavoidable defect of spiral arrangements is the necessity for moveable indices that require special adjustments and give but one result at one time.

The common slide rule, when set to fulfil a given condition, fulfils it completely. Thus, by setting $\frac{1}{2}$, we have also $\frac{2}{4}$, $\frac{3}{6}$, $\frac{1234}{2468}$, &c., the whole range of such values being immediately visible.

To preserve this most important property—the essential property of the sliding rule—I have had one constructed in parts, yet so as to be continuous, and to afford a full scale in any position of the slide.

This instrument, of a convenient size, has a scale of ten feet only. It can be read exactly for three figures throughout, and for a fourth figure either exactly or nearly.

The applications of such a rule are manifold, and as the adjustments can only be taught, rule in hand, further explanation need not in this place be offered. A peculiar advantage is, that for special purposes, special slides can be made, and the working be thus rendered more easy.

The data of the following case of distributive proportion are not imaginary. The work was completed, with only two settings, in six minutes. The results are the same as found on Fuller's spiral scale.

A	B	C
227·5	149·9	40·2
243	160·1	42·7
109·5	72·1	19·3
309	203·6	54·6
42·5	28	7·5
108·5	71·4	19·1
325·5	214·4	57·5
20·5	13·5	3·6
516·5	340·2	91·2
474	312·3	83·7
53	34·9	9·4
25·5	14·8	3·9
98	64·6	17·3
2550·0	1679·8	450·0

Herein A is the standard column, and columns B and C are in the same proportion.

The rule settings are $\frac{168}{255}$ and $\frac{45}{255}$, the results are obtained by reference to the rule with the numbers in column A.

There are many other forms of the slide rule. My purpose is not to describe any, but to point out their general utility for actuarial calculations.

TABLE OF VALUES *of* $Log_{10}(1+i)$ *to* 15 *places, from* $100i=0$ *to* 10, *proceeding by* $\frac{1}{16}$*ths.*

(Specially computed for this Work by Mr. PETER GRAY.)

$100i$	$\log_{10}(1+i)$	$100i$	$\log_{10}(1+i)$
0	·000 000 000 000 000	3	·012 837 224 705 172
$\frac{1}{16}$	·000 271 349 263 375	$\frac{1}{16}$	·013 100 672 988 594
$\frac{2}{16}$	·000 542 529 092 294	$\frac{2}{16}$	·013 363 961 557 981
$\frac{3}{16}$	·000 813 539 698 220	$\frac{3}{16}$	·013 627 090 606 869
$\frac{4}{16}$	·001 084 381 292 220	$\frac{4}{16}$	·013 890 060 328 438
$\frac{5}{16}$	·001 355 054 084 966	$\frac{5}{16}$	·014 152 870 915 522
$\frac{6}{16}$	·001 625 558 286 737	$\frac{6}{16}$	·014 415 522 560 603
$\frac{7}{16}$	·001 895 894 107 420	$\frac{7}{16}$	·014 678 015 455 813
$\frac{1}{2}$	·002 166 061 756 507	3 $\frac{1}{2}$	·014 940 349 792 936
$\frac{9}{16}$	·002 336 061 443 105	$\frac{9}{16}$	·015 202 525 763 412
$\frac{10}{16}$	·002 705 893 375 925	$\frac{10}{16}$	·015 464 543 558 330
$\frac{11}{16}$	·002 975 557 763 293	$\frac{11}{16}$	·015 726 403 368 436
$\frac{12}{16}$	·003 245 054 813 147	$\frac{12}{16}$	·015 988 105 384 130
$\frac{13}{16}$	·003 514 384 733 037	$\frac{13}{16}$	·016 249 649 795 469
$\frac{14}{16}$	·003 783 547 730 127	$\frac{14}{16}$	·016 511 036 792 167
$\frac{15}{16}$	·004 052 544 011 197	$\frac{15}{16}$	·016 772 266 563 594
1	·004 321 373 782 642	4	·017 033 339 298 780
$\frac{1}{16}$	·004 590 037 250 476	$\frac{1}{16}$	·017 294 255 186 414
$\frac{2}{16}$	·004 858 534 620 328	$\frac{2}{16}$	·017 555 014 414 844
$\frac{3}{16}$	·005 126 866 097 449	$\frac{3}{16}$	·017 815 617 172 080
$\frac{4}{16}$	·005 395 031 886 706	$\frac{4}{16}$	·018 076 063 645 795
$\frac{5}{16}$	·005 663 032 192 590	$\frac{5}{16}$	·018 336 354 023 322
$\frac{6}{16}$	·005 930 867 219 212	$\frac{6}{16}$	·018 596 488 491 658
$\frac{7}{16}$	·006 198 537 170 307	$\frac{7}{16}$	·018 856 467 237 466
1 $\frac{1}{2}$	·006 466 042 249 232	4 $\frac{1}{2}$	·019 116 290 447 073
$\frac{9}{16}$	·006 733 382 658 968	$\frac{9}{16}$	·019 375 958 306 470
$\frac{10}{16}$	·007 000 558 602 124	$\frac{10}{16}$	·019 635 471 001 316
$\frac{11}{16}$	·007 267 570 280 934	$\frac{11}{16}$	·019 894 828 716 939
$\frac{12}{16}$	·007 534 417 897 257	$\frac{12}{16}$	·020 154 031 638 333
$\frac{13}{16}$	·007 801 101 652 584	$\frac{13}{16}$	·020 413 079 950 161
$\frac{14}{16}$	·008 067 621 748 033	$\frac{14}{16}$	·020 671 973 836 756
$\frac{15}{16}$	·008 333 978 384 351	$\frac{15}{16}$	·020 930 713 482 124
2	·008 600 171 761 917	5	·021 189 299 069 938
$\frac{1}{16}$	·008 866 202 080 743	$\frac{1}{16}$	·021 447 730 783 546
$\frac{2}{16}$	·009 132 069 540 472	$\frac{2}{16}$	·021 706 008 805 968
$\frac{3}{16}$	·009 397 774 340 380	$\frac{3}{16}$	·021 964 133 319 899
$\frac{4}{16}$	·009 663 316 679 379	$\frac{4}{16}$	·022 222 104 507 706
$\frac{5}{16}$	·009 928 696 756 016	$\frac{5}{16}$	·022 479 922 551 432
$\frac{6}{16}$	·010 193 914 768 475	$\frac{6}{16}$	·022 737 587 632 799
$\frac{7}{16}$	·010 458 970 914 574	$\frac{7}{16}$	·022 995 099 933 200
2 $\frac{1}{2}$	·010 723 865 391 773	5 $\frac{1}{2}$	·023 252 459 633 711
$\frac{9}{16}$	·010 988 598 397 168	$\frac{9}{16}$	·023 509 666 915 084
$\frac{10}{16}$	·011 253 170 127 497	$\frac{10}{16}$	·023 766 721 957 749
$\frac{11}{16}$	·011 517 580 779 137	$\frac{11}{16}$	·024 023 624 941 817
$\frac{12}{16}$	·011 781 830 548 107	$\frac{12}{16}$	·024 280 376 047 080
$\frac{13}{16}$	·012 045 919 630 068	$\frac{13}{16}$	·024 536 975 453 010
$\frac{14}{16}$	·012 309 848 220 326	$\frac{14}{16}$	·024 793 423 338 763
$\frac{15}{16}$	·012 573 616 513 829	$\frac{15}{16}$	·025 049 719 883 176

TABLE OF VALUES—(*continued*).

$100i$	$\log_{10}(1+i)$	$100i$	$\log_{10}(1+i)$
6	·025 305 865 264 770	8	·033 423 755 486 949
$\frac{1}{16}$	·025 561 859 661 751	$\frac{1}{16}$	·033 675 010 617 998
$\frac{2}{16}$	·025 817 703 252 009	$\frac{2}{16}$	·033 926 120 472 870
$\frac{3}{16}$	·026 073 396 213 121	$\frac{3}{16}$	·034 177 085 219 469
$\frac{4}{16}$	·026 328 938 722 349	$\frac{4}{16}$	·034 427 905 025 403
$\frac{5}{16}$	·026 584 330 956 644	$\frac{5}{16}$	·034 678 580 057 992
$\frac{6}{16}$	·026 839 573 092 644	$\frac{6}{16}$	·034 929 110 484 266
$\frac{7}{16}$	·027 094 665 306 676	$\frac{7}{16}$	·035 179 496 470 968
6 $\frac{1}{2}$	·027 349 607 774 756	8 $\frac{1}{2}$	·035 429 738 184 548
$\frac{9}{16}$	·027 604 400 672 592	$\frac{9}{16}$	·035 679 835 791 174
$\frac{10}{16}$	·027 859 044 175 579	$\frac{10}{16}$	·035 929 789 456 723
$\frac{11}{16}$	·028 113 538 458 809	$\frac{11}{16}$	·036 179 599 346 787
$\frac{12}{16}$	·028 367 883 697 061	$\frac{12}{16}$	·036 429 265 626 675
$\frac{13}{16}$	·028 622 080 064 812	$\frac{13}{16}$	·036 678 788 461 406
$\frac{14}{16}$	·028 876 127 736 229	$\frac{14}{16}$	·036 928 168 015 719
$\frac{15}{16}$	·029 130 026 885 175	$\frac{15}{16}$	·037 177 404 454 068
7	·029 383 777 685 209	9	·037 426 497 940 623
$\frac{1}{16}$	·029 637 380 309 585	$\frac{1}{16}$	·037 675 448 639 274
$\frac{2}{16}$	·029 890 834 931 254	$\frac{2}{16}$	·037 924 256 713 626
$\frac{3}{16}$	·030 144 141 722 864	$\frac{3}{16}$	·038 172 922 327 006
$\frac{4}{16}$	·030 397 300 856 762	$\frac{4}{16}$	·038 421 445 642 459
$\frac{5}{16}$	·030 650 312 504 992	$\frac{5}{16}$	·038 669 826 822 752
$\frac{6}{16}$	·030 903 176 839 298	$\frac{6}{16}$	·038 918 066 030 369
$\frac{7}{16}$	·031 155 894 031 127	$\frac{7}{16}$	·039 166 163 427 521
7 $\frac{1}{2}$	·031 408 464 251 624	9 $\frac{1}{2}$	·039 414 119 176 137
$\frac{9}{16}$	·031 660 887 671 635	$\frac{9}{16}$	·039 661 933 437 870
$\frac{10}{16}$	·031 913 164 461 711	$\frac{10}{16}$	·039 909 606 374 097
$\frac{11}{16}$	·032 165 294 792 103	$\frac{11}{16}$	·040 157 138 145 918
$\frac{12}{16}$	·032 417 278 832 769	$\frac{12}{16}$	·040 404 528 914 159
$\frac{13}{16}$	·032 669 116 753 368	$\frac{13}{16}$	·040 651 778 839 370
$\frac{14}{16}$	·032 920 808 723 266	$\frac{14}{16}$	·040 898 888 081 828
$\frac{15}{16}$	·033 172 354 911 534	$\frac{15}{16}$	·041 145 856 801 536
		10	·041 392 685 158 225

Mr. Gray has furnished the following interesting note on the calculations:—

"The principle I have used is the following: If of two series, the terms composing the one are each n times the terms composing the other, then the logs of the terms of both series will have the same series of finite differences. If, therefore, the logs of the terms of one of the series are known, those of the terms of the other can readily be formed.

"The logs wanted are those of 10000, 1000625, 100125, and so on. Multiplying these terms by 16, we have 1600, 1601, 1602 . . . . 1760, 160 terms in all. The logs of these are to be found in column 1 of my

24 log tract to the required extent—say 15 places.* Well, writing those logs on alternate lines, and their differences on the intermediate lines, those differences will be the differences of the required series. And carrying them out into a parallel column, and adding in the usual way, with 0 (=log 100 . . .) for the initial term, we have the required series of logs.

"I have checked the work thoroughly as I proceeded, and I believe every figure can be depended on, with one exception. Every 8th value is to be found in column 1 of my table aforesaid. Thus:

½	per-cent	opposite	·005
1	,,	,,	·010
1½	,,	,,	·015
2	,,	,,	·020 and so on.

"I have compared my results carefully with these, and in about half of them I find a discrepancy of a unit in the last place. I can see the cause of this. In fact, I anticipated it, although I am not prepared at this moment to formulate it."

* The work here referred to is Mr. Gray's "Tables for the formation of Logarithms and Anti-Logarithms to 24 or any less number of places." Laytons, 1876.

EXAMPLES FOR THE STUDENT ON THE FIRST TWO CHAPTERS.

CHAPTER I.

(1).—Find the true discount on 385·55 due 43 days hence, interest at 8 per-cent.

Ans.: 3·480.

(2).—The rate of interest being 5 per-cent payable half-yearly, and the income tax $\frac{7}{240}$ per 1, find the sum which in two years will amount to 1000.

Ans.: 908·533.

(3).—Show that 1 will amount at the end of 87 years to 1803·455, 2119·505, 2305·867, 2514·929, according as the nominal rate of interest, 9 per-cent, is convertible yearly, half-yearly, quarterly, or momently.

(4).—A person lends at the end of one year 1, at the end of two years 4, at the end of three years 9, &c., at simple interest. What will be the amount of the debt at the end of n years, interest at the rate i?

Ans.: $\dfrac{n.\overline{n+1}.\overline{2n+1}}{6} + \dfrac{n^2(n+1)(n-1)}{12}\,i.$

(5).—Show that the discount is half the harmonic mean between the sum due and the interest on it.

(6).—What sum will amount to 1 in 20 years, interest at the nominal rate of 5 per-cent convertible momently?

Ans.: ϵ^{-1}.

(7).—If a sum of money at a given rate of interest accumulate to p times its original amount in n years, and to p' times its original amount in n' years, show that $n'=n\log_p p'$.

(8).—If P represent the population of any place at a certain time, and every year the number of deaths is $\frac{1}{p}$th, and the number of births $\frac{1}{q}$th, of the whole population at the beginning of that year, required the population at the end of n years from that date.

$$\textit{Ans.}: \ \mathrm{P}\left(1-\frac{1}{p}+\frac{1}{q}\right)^n.$$

(9).—If a quantity change continuously in value from a to b in a given time t_1, the increase at any instant bearing a constant ratio to its value at that instant, prove that its value at any time t will be $a\left(\frac{b}{a}\right)^{\frac{t}{t_1}}$.

Chapter II.

(1).—Show that the true discount on 1 due n years hence, interest at the rate i, is equal to the value of an annuity of i to run for n years.

(2).—A sum of 84,000 is borrowed at $4\frac{3}{4}$ per-cent, to be repaid by 37 annual and equal payments. What will be the amount of each payment, and what portions of the seventh payment will be on account of interest and repayment of debt respectively?

(3).—Given that the values of $\log \frac{1}{a_{\overline{23}|}}$, when the rate of interest is $4\frac{3}{8}$ per-cent, $4\frac{1}{2}$ per-cent, or $4\frac{5}{8}$ per-cent respectively, are $\bar{2}\cdot8440512$, $\bar{2}\cdot8493118$, or $\bar{2}\cdot8545404$; find by interpolation the value of $\frac{1}{a_{\overline{23}|}}$, when the rate of interest is $4\frac{7}{16}$ per-cent.

Ans.: $\cdot070256$.

(4).—What is the present value of the lease, to run for 38 years, of an estate producing a clear income of 560, the sum

invested being assumed to bear interest throughout at $4\frac{3}{4}$ per-cent, and the portion of the annual income invested to reproduce the capital at the end of the term being assumed to bear interest at $3\frac{1}{4}$ per-cent?

Ans.: 9149·66.

(5).—If n be very large, and i the rate of interest, show that approximately

$$i=\frac{1}{a_{\overline{n}|}}-\frac{1}{a_{\overline{n}|}}\left(1+\frac{1}{a_{\overline{n}|}}\right)^{-n}.$$

Given that $317.a_{\overline{120}|}=6648{\cdot}223$, show that the rate of interest involved is $4\frac{3}{4}$ per-cent.

(6).—What would be the value of an annuity-certain for n years, first payment due six months hence?

Ans.: $(1+i)^{\frac{1}{2}}a_{\overline{n}|}$.

(7).—A and B are put in possession, in equal shares, of an annuity-certain for $2n$ years. They arrange to take the payments alternately, A taking the first. How much ought he to pay to B for the advantage he thus receives?

Ans.: $\dfrac{v(1-v^{2n})}{2(1+v)}$.

(8).—Find the value of an annuity-certain for n years whose several payments are 1, 2, 3, . . n.

Ans.: $\dfrac{\dfrac{1-v^n}{iv}-nv^n}{i}$.

(9).—Given 6·4012, the annuity which will liquidate a debt of 100 in 25 years, interest at 4 per-cent, find the annual sum payable in advance which will amount to 100 at the end of 25 years (25 payments).

Ans.: 2·308843.

(10).—Find what accumulative sinking fund per-cent will repay a loan in 20 years at 5 per-cent interest, the amount to which 1 will accumulate in 20 years at that rate being 2·653.

Ans.: 3·02568.

(11).—The present value of A per annum for n years certain is equal to the amount of an annuity of B per annum for n years certain; find the present value of 1 due n years hence.

(12).—Show that if there be two annuities, one payable at the end of every mth interval of a year, and the other at the end of every pth interval of a year, the amounts payable annually, the effective rate of interest, and the time for which they are to run being the same, the ratio of the value of the first annuity to that of the second is independent of the time for which they are to run, and is equal to $\frac{p}{m}\frac{(1+i)^{\frac{1}{p}}-1}{(1+i)^{\frac{1}{m}}-1}$.

(13).—If an annuity of 845 payable half-yearly be required to be converted into an annuity payable quarterly, the effective rate of interest being $3\frac{3}{4}$ per-cent, what sum should be paid per annum instead of 845?

Ans.: 841·1114.

(14).—If i be the nominal rate of interest convertible m times a year, and an annuity of 1 be payable k times a year, and a denotes its present value for n years, then if n be given by

$$n=\frac{-\log(1-aip)}{m\log\left(1+\frac{i}{m}\right)},\text{ show that } p=\frac{\left(1+\frac{i}{m}\right)^{\frac{m}{k}}-1}{\frac{i}{k}}.$$

(15).—In what time would 26 per month discharge a debt of 5700, the nominal rate of interest being $4\frac{1}{2}$ per-cent?

Ans.: 37·212 years.

(16).—Show if two sums of the same amount be payable, one n years hence, and the other $n+t$ years hence, then the "equated time" is less than $\frac{1}{2}(n+n+t)$, and is the same as the "equated time" for two annuities of equal amount to run for n and $n+t$ years respectively.

(17).—Which is the greater, $a_{\overline{n}|}+a_{\overline{n+t}|}$ or $2a_{\overline{(n+n+t)\frac{1}{2}}|}$?

(18).—A person borrows a sum of money and pays off at the end of each year as much of the principal as he pays interest for that year: find how much he owes at the end of n years.

(19).—What is the present value of an annuity which is to commence at the end of p years and to continue for ever, each payment being m times the preceding? What limitation is there as to m?

$$\textit{Ans.}: \ \frac{1}{(1+i)^{p-1}} \cdot \frac{1}{(1+i)-m}, \qquad m<(1+i).$$

(20).—If two joint proprietors have an equal interest in a freehold estate worth p per annum, but one of them purchase the the whole to himself by allowing the other an equivalent annuity of q for n years, find the relation between p and q.

$$\textit{Ans.}: \ q=\tfrac{1}{2}\frac{p}{1-v^n}.$$

(21).—Explain verbally the meaning of the equality

$$\left(i+\frac{1}{n}\right)v+\left\{i\left(1-\frac{1}{n}\right)+\frac{1}{n}\right\}v^2+\ldots+\left\{i\left(1-\frac{n-1}{n}\right)+\frac{1}{n}\right\}v^n=1,$$

and prove it.

(22).—If $P=Xa_{\overline{n}|}$ and $P+Q=Xa_{2n}$, then

$$X=\frac{P^2}{P-Q}\left\{\left(\frac{P}{Q}\right)^{\frac{1}{n}}-1\right\}.$$

(23).—A has just purchased an annuity for ever, and B, *with the same capital*, one for three years, when an income tax for three years is imposed. If the tax be 3 per-cent on the perpetual annuity, what ought it to be on B's annuity, if the value of both properties be taxed alike, the annuities being calculated with interest at 5 per-cent?

(24).—If p years' purchase must be paid for an annuity to continue for a certain number of years, and q years' purchase for an annuity to continue twice as long, determine the rate of interest.

$$\textit{Ans.}: \ i=\frac{2p-q}{p^2}.$$

(25).—A debenture of 100, bearing interest at the rate of 6 per-cent per annum and redeemable in 20 years, is purchased for 107·5: show how to determine approximately the rate of interest realized—given that at 5 per-cent the debenture would be worth 112·463.

Ans.: 5·379 per-cent.

(26).—Find the value of an annuity to run for n years after t years, that is an annuity for n years deferred t years, with the condition that the purchaser is to receive on his capital interest for the entire term at the rate i', and the sinking fund for accumulation is to bear interest at the rate i.

Why does not the ordinary formula $a_{\overline{n+t}|}-a_{\overline{t}|}$ (Art. 31) apply?

(27).—Show that $\dfrac{P_{\overline{n}|}}{P_{\overline{n}|}+i}=v^n$.

Give a verbal interpretation of this result.

(28).—Show that $n=\dfrac{\log\left(1+\dfrac{i}{P_{\overline{n}|}}\right)}{\log(1+i)}$.

(29).—What does the formula $a_{\overline{n}|}=\dfrac{1}{P_{\overline{n}|}+i}$ (Art. 26) become when annuity is payable and interest at the nominal rate i convertible m times a year?

What is then the value of $\dfrac{1}{a_{\overline{n}|}}-P_{\overline{n}|}$?

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